On the Existence and Convergence of the Markovian Traffic Equilibrium∗

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Abstract

Stochastic user equilibrium models are fundamental to the analysis of transportation systems. Equilibrium flows are typically obtained by a sequential approach: first route choice sets are generated, and then the demand is assigned to these routes with a route choice model. A class of link based models under a Markovian assumption on the choice behavior of the users has been proposed to deal with the drawbacks of such route based choice models. The idea has important advantages, such as relaxing the predetermined route choice set assumption and alleviating the need for route generation. However, despite its advantages, the idea has attracted lesser attention than the route based models, and thus far its implementation has been limited to the multinomial logit model. This is due to the additional complexity introduced by the dynamic choice process. Although these issues are discussed in literature, the discussion is scattered among various publications and lacks a thorough theoretical analysis to be useful beyond the computational studies given in these few publications. In this paper, we provide necessary and sufficient existence condition of the Markovian traffic assignment. We propose an algorithm to obtain the flows under general random utility maximization, and provide the complexity analysis of Markovian stochastic network loading mechanism. We show theoretically and demonstrate on small networks that Markovian assignment

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can be extremely inefficient and unrealistic unless the parameters of the model are chosen carefully. In order to deal with these issues and improve the practicality of Markovian choice models, we propose an approach to restrict the parameter space. This guarantees efficient convergence and realistic traffic assignment. In our experiments, we show that the proposed approach allows to obtain the Markovian traffic equilibrium flows efficiently for real sized traffic networks.

**Keywords:** Markovian traffic equilibrium, stochastic network loading, existence of equilibrium, convergence analysis

**1 Introduction**

Traffic equilibrium models are fundamental to the analysis of transportation systems. The inputs to such models are the network topology, origin-destination (OD) pairs, demands of OD pairs and link performance functions. The goal is to estimate the equilibrium link flows. Daganzo and Sheffi (1977) relaxed the perfect information assumption of the user equilibrium (Wardrop, 1952) and proposed a stochastic user equilibrium (SUE) with a discrete choice model to capture randomness in user route choice behavior.

Traffic networks are usually cyclic and include infinitely many routes. Traffic assignment models therefore follow a sequential approach. First, a set of eligible routes are generated with respect to some criteria. Then, equilibrium flows are obtained by assigning the OD demand probabilistically to these routes. Bekhor et al. (2006) provide an evaluation of choice set generation algorithms. These algorithms fall in the class of deterministic methods and ignore the effect of congestion. While there has been much focus on developing route based SUE models integrated with discrete choice models to accurately model user behavior, lesser attention has been paid to drawbacks of such route based models.

One approach that has been proposed in this context is to assume a sequential Markovian decision-making process where the route choice of a user is determined by successive link choices that is made independently of how the users reached the current node (Akamatsu, 1996, 1997; Bell, 1995; Baillon and Cominetti, 2008; Fosgerau et al., 2013; Mai et al., 2015). At a particular node, the users observe the instantaneous costs of the emanating links and make their choices by considering the expected costs from the head nodes to the destination. Evaluating the steady state flows in this Markov chain then allows the system planner to develop models that evaluate all routes without explicitly generating them. Despite its obvious advantages,
this model has not been as popular as the route based models. The reason is likely the additional complexity of the Markovian choice model that can be explained in three-folds.

1. An efficient method to calculate the expected perceived travel costs is required. In general, this calculation is nontrivial, and applications has been limited to the multinomial logit model thus far.

2. Markovian choice model requires solving a recursive system of nonlinear equations. The solution to this system might not exist for certain parameter settings. Furthermore, unlike route based models, convergence rate for solving this system significantly depends on the distribution of the error terms.

3. The model might lead to an extremely unrealistic traffic assignment. In particular, expected cost from a node to the destination might become negative. This leads to a traffic assignment with a high fraction of users choosing cyclic routes, i.e., routes that include cycles. In real life, users might follow cyclic routes; however, it is expected that choice probability of such routes are sufficiently small.

In a recent study, we focused on the first challenge (Ahipaşaoğlu et al., 2017). In particular, we proposed a distributionally robust Markovian traffic equilibrium. Proposed choice model, termed as the Marginal Markovian Distribution Model (M-MDM), assumes that the joint distribution of the error terms is not known, rather uses the information of marginal distributions. Due to the convexity of the formulation, M-MDM can be considered as an efficient alternative to the independently and identically distributed (iid) Gumbel error term assumption, which leads to Markovian logit (Akamatsu, 1996) or Recursive Logit (Fosgerau et al., 2013) models.

In this study, we focus on the second and third challenges. These problems were discussed by several authors for the special case where the error terms are iid Gumbel; however, the literature lacks a systematic approach to deal with them. Bell (1995) discussed the inefficiency of the solution algorithm and stated that a significant fraction of users might follow cyclic routes. The author suggested to increase the value of the dispersion parameter to make the users more cost sensitive. Akamatsu (1996) pointed out the possibility that the link flows become infinity, where the users keep cycling and never reach their destinations. Wong (1999) stated that the system converges unconditionally for acyclic networks, but not necessarily does so for general networks. In a recent study, Fosgerau et al. (2013) proposed an exponential transformation and a matrix inverse operation to solve
the Markovian system under the logit assumption. The authors also stated that the matrix might be ill conditioned or even singular. In this study, we formally state which conditions lead to these issues and propose a practical solution to circumvent these issues of the Markovian choice model for general error distributions.

In order to illustrate the challenges, consider the four-node network in Figure 1 taken from [Akamatsu (1996)](1996) with a single OD pair (1, 4), where the numbers next to the arcs represent the deterministic component of the link costs. Let \( p_{ij}^d \) be the probability that a user at node \( i \) chooses link \((i, j)\) towards destination \( d \), and \( f_{ij} \) be the flow on link \((i, j)\). Finally, \( p_r \) represents the choice probability of route \( r \), calculated using link choice probabilities. Assume that the OD demand is 100 units. Table 1 represents different Markovian assignments for four cases, each corresponding to a different choice model setting. In particular, the first three cases are Recursive Logit models with different assumptions on the dispersion, and the fourth case is a Marginal Markovian Distribution Model with exponential marginals. Details of the choice models will be presented in Section 4.1. In Case 1 and Case 2, we observe traffic assignments where 2% and 14% of the commuters follow cyclic routes, respectively. In Case 3, total choice probability of direct routes (1-2-4, 1-2-3-4, 1-3-4 and 1-3-2-4) drops to only 18%. This results in a flow of around 470 units on each link of the cycle (2-3 and 3-2), while the total demand is only 100 units. Case 4 is the extreme case where the choice probability of the cycle is one \( (P_c^d = p_{23}^4 \times p_{32}^4 = 1) \), and hence, none of the users leaving node 1 reaches the destination. In other words, the solution does not exist. Note also that when the assignment becomes unrealistic, the number of iterations to solve the problem increases rapidly. These cases indicate that the Markovian assignment can be reasonable or extremely unrealistic; moreover, the convergence can be efficient or extremely slow even in small networks.

In this paper, we provide theoretical results for the existence and convergence of the Markovian assignment in order to deal with these issues, and to guarantee both a realistic and a computationally efficient assignment. Contributions of the paper can be summarized as follows:

(a) We provide the necessary and sufficient existence condition for the Markovian traffic assignment. We show that the condition depends only on the network topology and expected perceived travel costs, and hence, existence can easily be tested.

(b) We propose a general solution algorithm for the nonlinear system of equations of the Markovian choice model. The algorithm is basically
We provide the complexity analysis. Furthermore, we show both theoretically and by our experiments that the convergence is efficient whenever the traffic assignment is realistic.

(c) We provide sufficient conditions on the parameter space to guarantee nonnegative expected costs. These conditions bound the cyclic route choice probabilities and prevent unrealistic assignment. Furthermore, bounds on cyclic flows improve the convergence performance.

(d) Taking congestion into account, we extend aforementioned results to the Markovian traffic equilibrium.

The outline of the paper is as follows. In Section 2, we review stochastic network loading mechanism for the general route based and Markovian choice models. In Section 3, we propose the necessary and sufficient existence condition for the Markovian assignment. In Section 4, we discuss the general solution algorithm and provide its convergence analysis. In Section 5, we focus on restricting the parameter space. We propose sufficient conditions to guarantee realistic flows and computational efficiency. In Section 6, we extend the results to the Markovian traffic equilibrium. We present our computational results in Section 7 and concluding remarks in Section 8.

2 Stochastic Network Loading

Stochastic network loading (SNL) mechanism loads the network while the dependence of link costs on link flows is not considered (Sheffi 1985). SNL is usually used as the intermediate step in SUE solution approaches, such
Table 1: Example Markovian assignments for the four-node network.

<table>
<thead>
<tr>
<th></th>
<th>Case 1</th>
<th>Case 2</th>
<th>Case 3</th>
<th>Case 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f_{12}$ ($p_{412}$)</td>
<td>98.20 (0.98)</td>
<td>88.08 (0.88)</td>
<td>54.98 (0.55)</td>
<td>50.00 (0.50)</td>
</tr>
<tr>
<td>$f_{13}$ ($p_{413}$)</td>
<td>1.80 (0.02)</td>
<td>11.92 (0.12)</td>
<td>45.02 (0.45)</td>
<td>50.00 (0.50)</td>
</tr>
<tr>
<td>$f_{23}$ ($p_{423}$)</td>
<td>13.57 (0.14)</td>
<td>39.34 (0.37)</td>
<td>477.81 (0.90)</td>
<td>$\infty$ (1.00)</td>
</tr>
<tr>
<td>$f_{24}$ ($p_{424}$)</td>
<td>86.71 (0.86)</td>
<td>67.60 (0.63)</td>
<td>50.25 (0.10)</td>
<td>0.00 (0.00)</td>
</tr>
<tr>
<td>$f_{32}$ ($p_{432}$)</td>
<td>2.08 (0.14)</td>
<td>18.86 (0.37)</td>
<td>473.07 (0.90)</td>
<td>$\infty$ (1.00)</td>
</tr>
<tr>
<td>$f_{34}$ ($p_{434}$)</td>
<td>13.29 (0.86)</td>
<td>32.40 (0.63)</td>
<td>49.75 (0.10)</td>
<td>0.00 (0.00)</td>
</tr>
<tr>
<td>$p_{124}$</td>
<td>0.85</td>
<td>0.56</td>
<td>0.05</td>
<td>0.00</td>
</tr>
<tr>
<td>$p_{1234}$</td>
<td>0.11</td>
<td>0.20</td>
<td>0.05</td>
<td>0.00</td>
</tr>
<tr>
<td>$p_{134}$</td>
<td>0.02</td>
<td>0.08</td>
<td>0.04</td>
<td>0.00</td>
</tr>
<tr>
<td>$p_{1324}$</td>
<td>0.00</td>
<td>0.03</td>
<td>0.04</td>
<td>0.00</td>
</tr>
<tr>
<td>$P_e^d = p_{23}^d \times p_{32}^d$</td>
<td>0.02</td>
<td>0.14</td>
<td>0.82</td>
<td>1.00</td>
</tr>
<tr>
<td>Iterations</td>
<td>5</td>
<td>7</td>
<td>59</td>
<td>-</td>
</tr>
</tbody>
</table>

as the Method of Successive Averages (MSA) algorithm. Therefore, if the SNL fails to provide a solution at any step of the SUE algorithm, then the equilibrium flows cannot be obtained. In route based models, the SNL procedure distributes the demand of each OD pair to the predetermined routes and the resulting route flows give the link flows. In Markovian models, on the other hand, SNL procedure is applied to each destination instead of each OD pair due to the memoryless property; and the link flows are assigned without generating the routes. This provides a computational advantage since the number of OD pairs might be significantly larger than the number of destinations in large networks. On the other hand, the SNL steps become more complex in the Markovian context, as explained in the following subsections.

### 2.1 Route Based Choice Model

Consider a traffic network $G = (N,A)$ where $N$ is the set of nodes and $A$ is the set of links. Let $W$ be the set of OD pairs and $h_o^d > 0$ be the demand for OD pair $(o,d)$. For each OD pair $(o,d)$, a predetermined route set $R_{od}$ is given. The flow on route $r$ and on link $(i,j)$ are denoted by $x_r$ and $f_{ij}$, respectively; and $\delta_{ij,r}$ equals one if route $r$ includes link $(i,j)$, and zero otherwise. Choice probability of route $r$ is denoted by $p_r$ satisfying $\sum_{r \in R_{od}} p_r = 1$. Uncertainty is usually introduced by additive random utility terms, $\epsilon_r$. Deterministic travel cost of link $(i,j)$ is designated by $t_{ij}$ which is assumed to be positive and finite. For a given link cost vector $t = (t_{ij})_{(i,j) \in A}$,
the SNL steps can be defined as follows:

\[ p_r = \Pr \left( r \in \arg\min_{s \in R_{od}} \left\{ \sum_{(i,j) \in A} t_{ij} \delta_{ij,s} - \tilde{\epsilon}_s \right\} \right), \quad \forall r \in R_{od}, (o,d) \in W, \quad (1) \]

\[ x_r = h_o^d p_r, \quad \forall r \in R_{od}, (o,d) \in W, \quad (2) \]

\[ f_a = \sum_{(o,d) \in W} \sum_{r \in R_{od}} x_r \delta_{ij,r}, \quad \forall (i,j) \in A. \quad (3) \]

Note that the predetermined choice set provides a level of abstraction from the network structure. Considering that the linear transformations in steps (2)-(3) are computationally inexpensive, route based SNL can efficiently be solved provided that the choice probabilities can efficiently be calculated. Multinomial logit and weibit models that have closed-form choice probability expressions (Dial, 1971; Cascetta et al., 1996; Ben-Akiva and Bierlaire, 1999; Castillo et al., 2008; Fosgerau and Bierlaire, 2009; Kitthamkesorn and Chen, 2013; Nakayama and Chikaraishi, 2015; Xu et al., 2015), and Marginal Distribution Models (Ahipaşaoğlu et al., 2016) represented with a convex formulation can be considered in this class of choice models. Furthermore, under mild assumptions, these models guarantee existence of the SNL solution for any finite link cost vector. Therefore, the issues presented in Table 1 are not likely to be observed in route based models.

2.2 Markovian Choice Model

Unlike the route based models, the choice process in the Markovian setting is recursive. A user at a particular node chooses one of the links emanating from that node to move to the next state with a goal to reach the destination, and proceeds in this manner until she reaches the destination. We define \( G_d = (N_d \cup \{d\}, A_d) \) as the subnetwork that can be used by the users towards destination \( d \). Here, \( A_d \) is the subset of links \((i,j) \in A\) for which \( i \) is reachable from an origin \( o \) such that \((o,d) \in W\) without visiting \( d \), and destination \( d \) is reachable from \( j \). The subset \( N_d \) contains all tail nodes of links in \( A_d \), which are the transient states for destination \( d \). Finally, we let \( N_d^+(i) \) denote the out-neighbourhood of node \( i \) in \( G_d \). Without loss of generality, we assume that \( A = \cup_{d \in D} A_d \). The choice at a node is independent of the previous states the user visits. The uncertainty in the choice behavior is introduced by modelling the cost of link \((i,j)\) as a random variable expressed as \( t_{ij} - \tilde{\epsilon}_{ij}^d \) where \( \tilde{\epsilon}_{ij}^d \) is a random utility error term of using the link towards destination \( d \). The random term captures the components of the
preferences of the users that is unobservable to the system planner. Let \( w^d_j \) denote the expected minimum cost of travelling from node \( j \) to destination \( d \). Then, the Markovian SNL procedure can be summarized as follows:

\[
\begin{align*}
    w_i^d &= \varphi_i^d \left( t, w^d \right), & \forall i \in N_d, d \in D, \\
    w_d^d &= 0, & \forall d \in D, \\
    p_{ij}^d &= \Pr \left( j \in \arg\min_{k \in N^+_d(i)} \left\{ t_{ik} + w_k^d - \tilde{\epsilon}_{ik}^d \right\} \right), & \forall (i, j) \in A_d, d \in D, \\
    n^d &= \left[ I - Q_d \right]^{-1} h^d, & \forall d \in D, \\
    g_{ij}^d &= n_i^d p_{ij}^d, & \forall (i, j) \in A_d, d \in D, \\
    f_{ij} &= \sum_{d \in D: (i, j) \in A_d} g_{ij}^d, & \forall (i, j) \in A.
\end{align*}
\]

In equation (4), \( w_d^d = (w_i^d)_{i \in N_d} \) and the function \( \varphi_i^d \left( t, w^d \right) \) is the satisfaction function as defined in Sheffi (1985):

\[
\varphi_i^d \left( t, w^d \right) = \mathbb{E} \left[ \min_{j \in N^+_d(i)} \left\{ t_{ij} + w_j^d - \tilde{\epsilon}_{ij}^d \right\} \right], & \forall i \in N_d, d \in D.
\]

Equations (4)-(5) define a system of equations in variables \( w_i^d \). Once the expected costs solving this system are obtained, choice probabilities are calculated by (6). The square matrix \( Q_d \) is of size \( |N_d| \) and represents the transition matrix of the Markovian choice process for the transient states:

\[
(Q_d)_{ij} = \begin{cases} 
    p_{ij}^d, & \text{if } (i, j) \in A_d \text{ and } j \neq d, \\
    0, & \text{otherwise.}
\end{cases}
\]

In equation (7), \( I \) is the identity matrix, \( \tau \) is the transpose operator, and \( [I - Q_d]^{-1} \) is the fundamental matrix of the absorbing Markov chain, \((i,j)^{th}\) entry of which represents the expected number of times a user with origin \( i \) visits node \( j \) before reaching the destination. The \( i^{th} \) entry of the demand vector \( h^d \) equals \( h_i^d > 0 \) if \((i,d) \in W\) and zero otherwise. Then, the \( i^{th} \) entry of the vector \( n^d, n_i^d \), is the number of users leaving node \( i \) towards destination \( d \). These users are distributed to the emanating links with respect to the conditional link choice probabilities \( p_{ij}^d \) by equation (8), where \( g_{ij}^d \) is the number of users using link \((i,j)\) towards destination \( d \). Link flows, \( f_{ij} \), are calculated in (9). Finally, note that the route choice probabilities
can be obtained from link choice probabilities as follows:

\[ p_r = \prod_{(i,j) \in A} \left( p^d_{i,j} \right)^{\delta_{ij,r}}, \quad \forall r \in R^*_{od}, (o,d) \in W, \]  

where \( R^*_{od} \) is the universal set of all routes from \( o \) to \( d \), including cyclic routes.

In large traffic networks we generally have \( |W| \gg |D| \). However, Markovian SNL, requiring to solve two systems of equations, is apparently more complex than route based SNL. First system is defined by the equations (4)-(5). Complexity of solving this system is not only related to satisfaction function (10), but also to the network topology. This is due to the recursive decisions at the nodes and the fact that traffic networks are cyclic. Second, the linear system \( [I - Q^T_d] \mathbf{n}^d = \mathbf{h}^d \) needs to be solved for \( \mathbf{n}^d \) in (7). We discuss in the next section that existence of the solution to the first system guarantees nonsingularity of \( I - Q^T_d \), and hence, the system (4)-(5) plays a critical role in the existence of the Markovian assignment.

The notable work of Baillon and Cominetti (2008) proposed a Markovian traffic equilibrium (MTE) model for the congested case. The model provides a framework for the general error distributions. The authors propose an equivalent convex optimization formulation to obtain the equilibrium flows. Furthermore, the authors prove the existence of the MTE flows under mild assumptions on the error distribution in addition to the nonnegativity assumption on the expected costs, i.e., \( w^d_i \geq 0, \forall i \in N_d, d \in D \). We would like to note that nonnegativity of the expected costs is a sufficient existence condition. We demonstrate that the equilibrium with negative expected costs might exist. In other words, nonnegativity of expected costs implies existence, however the reverse does not necessarily hold. Furthermore, expected costs are not inputs of the assignment model, rather they are system variables. We relax this assumption on the system variables and propose necessary and sufficient existence condition that depends on distribution parameters and the network structure.

On the other hand, the case where the nodes have negative expected costs, while the cost at the destination is zero by the boundary condition, is unintuitive. Moreover, it results in an increase in the cyclic flows leading to an unrealistic traffic assignment. We also show that the complexity of solving the nonlinear system (4)-(5) depends directly on the choice probability of the cycles in the network. Keeping the expected costs nonnegative not only prevents unrealistic traffic assignment but also improves convergence performance. We propose sufficient conditions on the parameter space to achieve such a solution.
3 Existence of the Markovian Traffic Assignment

In this section, we present characteristics of the Markovian choice model with an objective to build the necessary and sufficient existence condition of the Markovian traffic assignment. As discussed in Sheffi (1985) and Baillon and Cominetti (2008), the satisfaction function (10) has the following important properties:

P1 $\phi^d_i(t, w^d_j)$ is componentwise increasing and concave in $t_{ij}$ and $w^d_j$ for all $j \in N^+_d(i), i \in N_d, d \in D$. Furthermore, $|\phi^d_i(t, w^d_j)| < \infty$ if and only if $t$ and $w^d$ are finite.

P2 $\frac{\partial \phi^d_i(t, w^d_j)}{\partial (t_{ij} + w^d_j)} = p^d_{ij} \in [0, 1], \forall (i, j) \in A_d, d \in D.$

Furthermore, $p^d_{ij} > 0$ if $t_{ij}$ and $w^d_j$ are finite for all $j \in N^+_d(i)$.

P3 $\phi^d_i(t, w^d) \leq \min_{j \in N^+_d(i)} \{t_{ij} + w^d_j - E[\tilde{\epsilon}^d_{ij}]\}, \forall i \in N_d, d \in D.$

We start by providing upper bounds on the expected costs that solve the Markovian subsystem (4)-(5).

**Lemma 1.** Let $s^d_{ij} := t_{ij} - E[\epsilon^d_{ij}]$ be the expected perceived cost of link $(i, j)$ for users with destination $d$, and $\pi^d_i$ be the shortest path distance from node $i$ to destination $d$ with respect to arc lengths $s^d_{ij}$. The expected minimum costs $w^d_i$ that solve the system (4)-(5) are bounded above by $\pi^d_i$:

$$w^d_i \leq \pi^d_i, \quad \forall i \in N_d, d \in D.$$  (12)

**Proof.** Consider a path $r \in R_{id}$ from $i \in N_d$ to $d$ following the sequence of nodes $r_1, r_2, \ldots, r_n$, where $r_1 = i$ and $r_n = d$. By the boundary condition (5), we have $w^d_{r_n} = w^d_d = 0$. Property $\text{P3}$ implies the following inequality:

$$w^d_{r_{n-1}} \leq t_{r_{n-1}r_n} - E[\tilde{\epsilon}^d_{r_{n-1}r_n}] = s^d_{r_{n-1}r_n}.$$

Using this bound with properties $\text{P1}$ and $\text{P3}$ we obtain the following inequality:

$$w^d_{r_{n-2}} \leq t_{r_{n-2}r_{n-1}} + w^d_{r_{n-1}} - E[\tilde{\epsilon}^d_{r_{n-2}r_{n-1}}]$$

$$\leq t_{r_{n-2}r_{n-1}} - E[\tilde{\epsilon}^d_{r_{n-2}r_{n-1}}] + s^d_{r_{n-1}r_n} = s^d_{r_{n-1}r_n} + s^d_{r_{n-2}r_{n-1}}.$$
Moving backwards in the same manner, we obtain
\[ w^d_i \leq \sum_{j=1}^{n-1} s_{r_j r_{j+1}}. \]
Right-hand-side of the inequality is the cost of path \( r \) with respect to arc lengths \( s_{ij}^d \). Since the inequality must hold for all paths in \( R_d \) including the shortest one, we have \( w^d_i \leq s^d_i \). This completes the proof.

In the following proposition, we show that the solution to the Markovian SNL system does not exist if the network contains at least one negative cost cycle with respect to expected perceived link costs \( s_{ij}^d \).

**Proposition 1.** Let \( G_d \) be the set of nodes that are included by at least one negative cost cycle in \( G_d \) with respect to expected perceived link costs \( s_{ij}^d \); and let \( T_d \) be the set of nodes that can reach a node in \( G_d \). If \( C_d \neq \emptyset \) for any destination \( d \in D \), the following statements hold:

(i) The system (4)-(5) diverges such that \( w^d_i = -\infty \) for all nodes \( i \in C_d \cup T_d \).

(ii) Choice probability of routes from any node in \( C_d \cup T_d \) to destination \( d \) is zero. Equivalently, \( p^d_{ij} = 0 \) if \( i \in C_d \cup T_d \) and \( j \notin C_d \cup T_d \).

(iii) The matrix \( I - Q_d \) is singular.

(iv) The flow on the links of the negative cost cycles blows up. In other words, we have \( n^d_i = \infty, \forall i \in C_d \) and \( g^d_{ij} = f_{ij} = \infty \) for all \((i,j)\) on a negative cost cycle.

Therefore, if there exists a negative cost cycle with respect to expected perceived link costs \( s_{ij}^d \) for at least one of the destinations, there is no solution to the Markovian SNL equations (4)-(9).

**Proof.** Having \( C_d \neq \emptyset \), we have \( s^d_i = -\infty, \forall i \in C_d \cup T_d \). Lemma 1 directly implies that \( w^d_i = -\infty, \forall i \in C_d \cup T_d \), and hence the system (4)-(5) does not have a solution.

For the second statement, consider a node \( i \in C_d \cup T_d \). Define \( M_d = N_d \setminus (C_d \cup T_d) \) as the set of nodes that cannot reach a negative cost cycle. By definition, there exists a simple path from all nodes in \( N_d \) to \( d \). Therefore, node \( i \) must have at least one simple path to \( d \), and at least one other path to a negative cost cycle since \( i \in C_d \cup T_d \). Equivalently, there must exist a node \( j \in C_d \cup T_d \) reachable from \( i \) with at least two arcs \((j,k)\) and \((j,l)\) such that \( k \in M_d \cup \{d\} \) and \( l \in C_d \cup T_d \). By the first statement, we have \( w^d_i = -\infty \). Therefore, the arc \((j,k)\) attracts zero probability by equation (6). This proves the second statement.
Aggregating the states in subsets $\mathcal{C}_d$, $\mathcal{T}_d$ and $\mathcal{M}_d$, the transition matrix $P_d$ and the matrix $I - Q_d$ can be expressed as follows:

$$P_d = \begin{bmatrix} 1 & 0 & 0 & 0 \\ >0 & 0 & 0 & 0 \\ 0 & 0 & 0 & >0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad I - Q_d = \begin{bmatrix} 0 & 0 & 0 \\ <0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$  

The matrix $I - Q_d$ is singular. The Markov chain no longer has a single absorbing state, instead there are two recurrent classes, $\mathcal{C}_d$ and $\mathcal{T}_d$. These users keep travelling within the cycle, and hence, the flow on the arcs of the negative cost cycles blows up. This proves the second and third statements. Note that the chain may be solved for the nodes in $\mathcal{C}_d \cup \mathcal{T}_d$ and $\mathcal{M}_d \cup \{d\}$ separately. However, the nodes in $\mathcal{C}_d \cup \mathcal{T}_d$ are not redundant by definition of $G_d$, i.e., there exists at least one node $i$ in $\mathcal{C}_d \cup \mathcal{T}_d$ such that $h_i^d > 0$. Therefore, there is no solution to the Markovian SNL equations (4)-(9).

Proposition 1 provides a sufficient condition for divergence. In the following lemma, we provide a monotonically nonincreasing series of expected costs that will be used in developing the necessary existence condition.

**Lemma 2.** Let $w_i^{d(n)}$, $n \in \{0, 1, \ldots \}$ be a series of real numbers for each node $i \in N_d$, and let $w_i^{d(n)} = \left( w_i^{d(n)} \right)_{i \in N_d}$. The series is defined as follows:

$$w_i^{d(0)} = s_i^d, \quad w_i^{d(n+1)} = \varphi_i^d (t, w_i^{d(n)}) \), \quad \forall n \in \{1, 2, \ldots \}, i \in N_d.$$

Then, the series is nonincreasing, i.e.,

$$w_i^{d(n)} \leq w_i^{d(n-1)}, \quad \forall n \in \{1, 2, \ldots \}, i \in N.$$

**Proof.** The monotonicity holds for destination since $w_d^{d(n)} = 0, \forall n$. For the transient nodes, we have $w_i^{d(1)} \leq s_i^d = w_i^{d(0)}, \forall i \in N_d$ by Lemma 1. We make the induction assumption that $w_i^{d(n)} \leq w_i^{d(n-1)}, \forall n \in \{1, \ldots, m-1\}, i \in$
This implies that \( t_{ij} + w_j^{d(m-1)} \leq t_{ij} + w_j^{d(m-2)}, \forall (i,j) \in A_d. \) By property \([P1]\) the monotonicity holds for \( m \) as follows:

\[
w_i^{d(m)} = \varphi_i^d \left( t, w^{d(m-1)} \right) \leq \varphi_i^d \left( t, w^{d(m-2)} \right) = w_i^{d(m-1)}, \quad \forall i \in N_d.
\]

Then, we state that the series is nonincreasing by induction. \( \square \)

The next proposition guarantees that Markovian SNL has a solution provided that there exists no negative cost cycle with respect to expected perceived link costs \( s_{dij} \). For the proof, we utilize a node labeling \([l], l \in \{1, 2, \ldots, |N_d| + 1\}\). If there exists an arc \((i, j)\) and if \( i \) is labeled prior to \( j \), then there must exist a path from \( j \) back to \( i \); or equivalently, \( i \) and \( j \) are included in at least one cycle. We define the subset \( T_l = \{1], [2], \ldots, [l]\} \) including the nodes with a label less than or equal to \( l \). Finally, we partition the head nodes of emanating links from node \([l]\) into two subsets as \( N_d^+(l) = D_l \cup C_l \), where \( D_l \) and \( C_l \) are defined as follows:

\[
D_l = N_d^+(l) \cap T_{l-1}, \quad \forall l \in \{2, 3, \ldots, |N_d| + 1\},
\]

\[
C_l = N_d^+(l) \setminus T_{l-1}, \quad \forall l \in \{2, 3, \ldots, |N_d| + 1\}.
\]

Among the emanating links from node \([l]\), the ones in \( D_l \) lead to nodes that are labeled prior to \([l]\), and the remaining ones in \( C_l \) head to nodes that are labeled after \([l]\). The labeling is characterized by the following conditions:

\[
[1] = d,
\]

\[
D_l \neq \emptyset, \quad \forall l \in \{2, 3, \ldots, |N_d| + 1\},
\]

\[
j \mapsto [l], \quad \forall j \in C_l, l \in \{2, 3, \ldots, |N_d| + 1\},
\]

where \( i \mapsto j \) denotes that there exists a path from node \( i \) to node \( j \). Node labels for the four-node network are presented in Table 2. As indicated above, \( D_l \) is nonempty for all nodes except for \([1]\) = \( d \); and there exists a path from all nodes in \( C_l \) back to node \([l]\). Finally note that \( T_{|N_d|+1} = N_d \cup \{d\} \).

It can be shown that such a labeling exists for any network \( G_d \), and we provide a simple algorithm to label the nodes in Appendix A.1.

**Proposition 2.** If there exists no negative cost cycle in \( G_d \) with respect to expected perceived link costs \( s_{dij} \) for any destination \( d \); or equivalently if \( C_d = \emptyset, \forall d \in D \), then there exists a unique solution to the Markovian SNL equations (4)-(9).
Proof. Existence. Lemma 1 provides finite upper bounds for the expected costs, and Lemma 2 defines a nonincreasing series of the expected costs. Thus, it is sufficient to show that the costs are bounded below to prove existence.

Consider the node labeling described above. We have $w_d[1] = w_d^d = 0$ by the boundary condition. We make the induction assumption that $w_d[l]$ is finite for $l = \{1, 2, \ldots, m - 1\}$. We investigate node $[m]$ in two cases. First, consider the case where $C_m = \emptyset$, and hence, $N_d^+(\{m\}) = D_m \subseteq T_{m-1}$. Note that $w_j^d$ is finite for all $j \in T_{m-1}$ by the induction assumption. Therefore, we have $t_{[m]j} + w_j^d > -\infty$ for all $j \in D_m$. Then, property P1 implies that $w_d^d[1]$ is also finite.

Second, consider the case where $C_m \neq \emptyset$. Let $q$ be a cycle including both node $[m]$ and some node $j \in C_m$. Let $N_q$ and $A_q$ be the set of nodes and arcs in cycle $q$, respectively. We show the finiteness of $w_d^d[1]$ by contradiction. We first assume $w_d^d[1] = -\infty$. Under this assumption, all nodes in the cycle must have unbounded expected costs, i.e., $w_i^d = -\infty$, $\forall i \in N_q$; and the arcs in the cycle must attract all flow with choice probability of one. Then, the expected cost expression in (10) for the nodes in the cycle reduces to the following:

$$w_d^d[i] = t_{ij} + w_j^d - E[\tilde{\epsilon}_{ij}] = s_{ij}^d + w_j^d, \quad \forall (i, j) \in A_q.$$ 

Substituting recursively, we obtain the following infinite series for $w_d^d$:

$$w_d^d[i] = \sum_{n=1}^{\infty} \sum_{(i,j)\in A_q} s_{ij}^d, \quad \forall i \in N_q.$$ 

Since there exists no negative cost cycle with respect to arc lengths $s_{ij}^d$, we have $\sum_{(i,j)\in A_q} s_{ij}^d \geq 0$. Thus, the assumption that $w_d^d[1] = -\infty$ cannot hold. By contradiction $w_d^d[1]$ is finite.
Since $w^d_{[m]}$ is finite in either of the cases, we conclude by induction that $w^d_i$ is finite for all $i \in N_d$. Therefore, there exists a solution to the subsystem (4)-(5). Furthermore, we have finite $t_{ij} + w^d_i$ for all links. By property $P2$ this implies that $p^d_{ij} > 0, \forall (i,j) \in A_d$. Thus, there exists a positive probability path $r$ such that $p_r = \prod_{(i,j) \in r} p^d_{ij} > 0$ from all nodes in $N_d$ to destination $d$. Then, the matrix $Q_d$ is substochastic and $I - Q_d$ is nonsingular. This guarantees the existence of the solution to the system (4)-(9).

**Uniqueness.** Suppose that $w^d$ and $\overline{w}^d$ are two finite solutions. We define $\delta^d_i = w^d_i - \overline{w}^d_i$ and $\delta = \max_{i \in N_d} \delta^d_i \geq 0$. Let node $i$ be a node where the maximum difference is attained. By definition of $\delta$, we have $\delta^d_j \leq \delta, \forall j \in N^+_d(i)$. Strict monotonicity of $w^d_i$ in $w^d_j$ (property $P1$) implies that $\delta^d_j = \delta, \forall j \in N^+_d(i)$. Note that there exists a simple path from every node $i \in N_d$ to destination $d$. Then, moving forward in the same manner we eventually reach the destination and show that $\delta_d = \delta$. Since $w^d_d = \overline{w}^d_d = 0$ by boundary condition (5), we have $\delta = 0$. This proves the uniqueness of the solution to the subsystem (4)-(5). Provided that there exists a finite solution, equations (6)-(9) provide a unique mapping from expected costs to link choice probabilities and link flows. Therefore, there exists at most one solution to the system (4)-(9).

Now, we are ready to provide the necessary and sufficient existence condition for the Markovian traffic assignment.

**Theorem 1.** The following statements are equivalent:

(i) There exists no negative cost cycle in $G_d$ with respect to expected perceived link costs $s^d_{ij}$ for any $d$, or equivalently $\mathcal{C}_d = \emptyset, \forall d \in D$.

(ii) There exists a unique solution to the system (4)-(5) for all destinations such that:

$$w^d_i \in \left(-\infty, \overline{s}^d_i\right], \forall i \in N_d, d \in D.$$ 

(iii) For each destination $d$, there exists at least one route $r$ from every node $i \in N_d$ to $d$ that attracts strictly positive choice probability, i.e.,

$$p_r = \prod_{(i,j) \in r} p^d_{ij} > 0.$$

(iv) The matrix $I - Q_d$ is nonsingular for all $d \in D$. 

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(v) There exists a unique solution to the Markovian SNL equations (4)-(9).

Proof. Equivalence of (i) and (ii) follows directly from Proposition 1 and Proposition 2. Having finite expected costs, property P2 guarantees that all links emanating from a node attract a strictly positive choice probability. Since destination $d$ is reachable from every node $i \in N_d$, there exists a positive probability path to the destination. On the other hand, if there exists no positive probability path from a particular node $i$, then there must be a cycle reachable from this node, all arcs of which have a probability of one. As discussed in the proof of Proposition 2, this might only happen if the expected costs of the nodes in the cycle are negative infinity. This proves the equivalence of (ii) and (iii). Having a positive probability path from every node $i \in N_d$ to $d$ guarantees that the Markov chain has a single recurrent class which includes only the destination. This guarantees that the matrix $I - Q_d$ is nonsingular. On the other hand, if the probability of reaching the absorbing state $d$ is zero from some node $i$, the matrix becomes singular as shown in the proof of Proposition 2. Therefore, statements (iii) and (iv) are equivalent. The equivalence of the existence of the solution to the Markovian SNL and nonsingularity of the matrix $I - Q_d$ is straightforward due to the direct mapping in steps (7)-(9). This completes the proof. 

Theorem 1 states that there exists a unique solution to the Markovian choice model if and only if there exists no negative cost cycle in $G_d$ with respect to expected perceived link costs $s_{ij}^d$. This condition can be tested easily using only the mean of the error terms. One simple method is to check whether the shortest path distance matrix obtained by the Floyd-Warshall algorithm (Floyd, 1962; Warshall, 1962) contains negative entries or not. Note that most discrete choice models assume a zero mean for the error terms, i.e., $E[\tilde{\epsilon}_{ij}^d] = 0$. Since we have $t_{ij} > 0, \forall (i,j) \in A$ by definition, we have $s_{ij}^d = t_{ij} - E[\tilde{\epsilon}_{ij}^d] > 0$ for these models. Therefore, an immediate corollary of Theorem 1 is that there exists a unique solution to such models. Consider, for instance, the Markovian model where the error terms $\tilde{\epsilon}_{ij}^d$ are iid Gumbel random variables with dispersion parameter $\theta_d^i > 0, \forall i \in N_d, d \in D$. The choice probability expression in (6) and the satisfaction function (10)
take the following forms:

\[
p_{ij}^d = \frac{\exp[-\theta_i^d (t_{ij} + w_j^d)]}{\sum_{k \in N_d^+(i)} \exp[-\theta_i^d (t_{ik} + w_k^d)]}, \quad \forall (i, j) \in A_d, d \in D, \quad (13)
\]

\[
\varphi_i^d (t, w^d) = -\frac{1}{\theta_i^d} \ln \left( \sum_{j \in N_d^+(i)} \exp[-\theta_i^d (t_{ij} + w_j^d)] \right), \quad \forall i \in N_d, d \in D.
\]

Means of the error terms are zero regardless of the value of the dispersion parameter. Then, there exists a solution to the Markovian logit or to the Recursive Logit model for any \( \theta_i^d > 0 \) by Theorem 1, and the system (4)-(5) converges unconditionally in both acyclic and cyclic networks. This seems contradictory to the convergence issues discussed in several studies. However, these problems stem from the machine precision and are due to the monotonic relationship between the expected costs and simple route choice probabilities as presented in the following proposition.

Let \( R_{id}^S \) be the set of all simple routes and \( R_{id}^C \) be the set of all cyclic routes from node \( i \) to \( d \), such that \( R_{id} = R_{id}^S \cup R_{id}^C \). Note that \( \sum_{r \in R_{id}^S} p_r + \sum_{r \in R_{id}^C} p_r = 1 \). In the next proposition, we show that sum of the choice probabilities of routes in \( R_{id}^S \) (respectively, in \( R_{id}^C \)) decreases (respectively, increases) as the expected costs decrease.

**Proposition 3.** Consider two different error distribution settings with equal means. Solutions to the system (4)-(5) for the two Markovian choice models are designated by \( w^d \) and \( \overline{w}^d \). Let \( p_r \) (respectively, \( p_{ij}^d \)) and \( \overline{p}_r \) (respectively, \( \overline{p}_{ij}^d \)) be route (respectively, link) choice probabilities corresponding to solutions \( w^d \) and \( \overline{w}^d \). If the expected costs of the first solution are componentwise less than or equal to those of the second solution, i.e.:

\[
w_i^d \leq \overline{w}_i^d, \quad \forall i \in N_d,
\]

then, the following holds:

\[
\sum_{r \in R_{id}^S} p_r \leq \sum_{r \in R_{id}^S} \overline{p}_r, \quad \forall i \in N_d.
\]

In other words, total choice probability of the simple paths of the solution with greater expected costs is greater than that with smaller expected costs.
Proof. If the network is acyclic, we have \( \sum_{r \in R_i} p_r = \sum_{r \in R_i} p_r = 1, \forall i \in N_d \), and hence, the statement holds. Now assume that the network is cyclic. Then, there must exists a node \( i^* \in N_d \) with \( |N_d^+(i^*)| \geq 2 \), such that there exists a head node \( j^* \in N_d^+(i^*) \) from which there exists only a single simple path, say \( r^* \), to the destination. Therefore, for this route \( r^* \), we have \( p_{ij} = p_{ij} = 1, \forall (i,j) \in r^* \). Consider the link choice at node \( i^* \) for the first solution \( w^d \). The deterministic cost of choosing link \((i^*,j^*)\) is \( t_{i^*j^*} + \sum_{(i,j) \in r^*} (t_{ij} - E[\tilde{\epsilon}_{ij}]) + w^d \), and that of choosing an alternative link \((i^*,j)\) is \( t_{i^*j} + w^d, \forall j \in N_d^+(i^*) \setminus \{j^*\} \). The deterministic cost of link \((i^*,j^*)\) for the second solution \( w^d \) remains the same since \( w^d = w^d = 0 \). On the other hand, deterministic cost of each alternative link \((i^*,j)\) is greater or equal to that of the first solution since \( w^d \leq w^d \). This implies that \( p_{ij} \leq p_{ij} \). This is sufficient to state that sum of simple route choice probabilities from node \( i^* \) to \( d \) increases as the expected costs increase. Since each simple or cyclic route from any node \( i \in N_d \) to \( d \) visits a node \( i^* \) as defined above with a probability of one, the inequality holds for all nodes.

Consider again the Markovian logit or the Recursive Logit model. As the dispersion parameter decreases, expected costs monotonically decrease by the expression \( [14] \) and property \( [11] \). By Proposition \( [3] \) this leads to a monotonic increase in the choice probabilities of the cyclic routes. Therefore, choice probability of some cycle becomes almost one for a sufficiently small dispersion parameter. Then, even though the matrix \( I - Q_d \) is theoretically nonsingular, its representation with the machine precision becomes singular. This points out the importance of limiting the parameter space to guarantee convergence, as well as to provide realistic traffic assignment where the cyclic flows are kept within reasonable limits.

4 Solution Algorithm

In literature, there exists solution algorithms for the Markovian logit or Recursive Logit models that assume iid Gumbel error terms. In this section, we present an algorithm to solve the Markovian choice model for the general error distributions. Proposed algorithm performs simple fixed-point iterations starting from the upper bounds given in Lemma \( [1] \) and generating the nonincreasing series defined in Lemma \( [2] \) to solve the subsystem \( (4)-(5) \). Once the expected costs are obtained, steps \( (6)-(9) \) are applied to obtain the link flows. The procedure is presented in Algorithm \( [1] \).

The complexity of the algorithm depends mainly on the inner loop in
Algorithm 1 Markovian Stochastic Network Loading Algorithm

Input: $t$ and $\xi$

1: procedure MSNL($t, \xi$)
2:     for each $d \in D$ do
3:         $w_i^{d(0)} = \pi_i^d$, $\forall i \in N_d$
4:         $w_d^{0} = 0$, $\forall n \in \{0, 1, \ldots\}$
5:     $n = 0$ and $error = \infty$
6:     while $error > \xi$ do
7:         $n = n + 1$
8:         for each $i \in N_d$ do
9:             $w_i^{d(n)} = \varphi_i^d (t, w_i^{d(n-1)})$
10:        end for
11:        $error = \max_{i \in N_d} \left\{ w_i^{d(n-1)} - w_i^{d(n)} \right\}$
12:     end while
13:     $w_i^d = w_i^{d(n)}$, $\forall i \in N_d$
14:     $p_{ij}^d = \Pr \left( j \in \arg\min_{k \in N_d^+ (i)} \left\{ t_{ik} + w_k^d - \bar{c}_{ik} \right\} \right)$, $\forall (i, j) \in A_d$
15:     $(Q_d)_{ij} = \begin{cases} p_{ij}^d, & \text{if } (i, j) \in A_d \text{ and } j \neq d, \\ 0, & \text{otherwise.} \end{cases}$
16:     $n^d = [I - Q_d^T]^{-1} h^d$
17:     $g_{ij}^d = n_i^d p_{ij}^d$, $\forall (i, j) \in A_d$
18:     end for
19:     $f_{ij} = \sum_{d \in D: (i, j) \in A_d} g_{ij}^d$, $\forall (i, j) \in A$
20: end procedure

Output: $f = (f_{ij})_{(i,j) \in A}$
lines 6-12 that solves the nonlinear subsystem (4)-(5), and on the linear system in line 16. Solving the linear system \( \mathbf{I} - \mathbf{Q}_d \mathbf{n}^d = \mathbf{h}^d \) might be considered computationally expensive. However, nodes on traffic networks usually have low degrees leading to a sparse matrix \( \mathbf{I} - \mathbf{Q}_d \). The sparsity allows to solve the system efficiently once its decomposition is obtained. Note that SNL is usually used as an intermediate step of the equilibrium algorithms and solved repeatedly, while the decomposition needs to be obtained once. Next, we provide the convergence analysis for the solution of the nonlinear system (4)-(5).

**Proposition 4.** Let \( Q^d_{i,i} \) be the set of all cycles from node \( i \) back to \( i \) that do not visit \( i \) or destination \( d \) as an intermediate node. Define \( P^d_{i,i} = \sum q \in Q^d_{i,i} p_q \) as the probability that a user at node \( i \) follows a cycle, where \( p_q = \prod_{(i,j) \in q} p^d_{ij} \) is the choice probability of cycle \( q \). Finally, let \( P^d_{c} \) be the maximum cycle probability for destination \( d \), defined as \( P^d_{c} = \max_{i \in N_d} P^d_{i,i} \). Then, the error of the Markovian SNL algorithm at step \( n \) is bounded above by:

\[
\max_{i \in N_d} \left\{ w^d_i(n) - w^d_i \right\} \leq \frac{(P^d_{c})^n}{1 - P^d_{c}} \max_{i \in N_d} \left\{ \left| w^d_i(n) - w^d_i(n-1) \right| \right\},
\]

where \( w^d_i, \forall i \in N_d \) is the solution of the system (4)-(5).

**Proof.** The proof follows from Banach’s fixed point theorem (Banach, 1922).

Let \( \Omega^d_{i} \) be the search region of the algorithm. Then, \( w^d_i(n) \in \Omega^d_{i} \) for any \( n \) by Lemma 1 and Lemma 2. We define the metric space \((\Omega^d, \eta)\) where \( \Omega^d = (\Omega^d_{i})_{i \in N_d} \) and

\[
\eta\left(w^d_i(n), w^d_i(n-1)\right) = \max_{i \in N_d} \left\{ \left| w^d_i(n) - w^d_i(n-1) \right| \right\}.
\]

The subsystem (4)-(5) can be represented as the mapping \( \varphi^d : \Omega^d \to \Omega^d \).

Rate of change in \( w^d_i(n) \) at point \( w^d_i(n) \) can be expressed by the first-order derivative that is obtained using property P2:

\[
\frac{\partial \varphi^d_i(t, w^d_i(n))}{\partial w^d_i(n)} = \sum_{j \in N^+_d(i)} p^d_{ij}(w^d_i(n)) \frac{\partial w^d_j(n)}{\partial w^d_i(n)},
\]

where \( p^d_{ij}(w^d_i(n)) \) is the choice probability calculated by (9) for the expected cost vector \( w^d_i(n) \). For an arbitrary node \( k \in N_d \), we have:

\[
\frac{\partial w^d_k(n)}{\partial w^d_i(n)} = \begin{cases} 
1, & \text{if } k = i, \\
0, & \text{if } k = d, \\
\sum_{l \in N^+_d(k)} p^d_{kl}(w^d_i(n)) \frac{\partial w^d_l(n)}{\partial w^d_i(n)}, & \text{otherwise.}
\end{cases}
\]
Substituting the derivatives successively in (16), we obtain:

\[ \frac{\partial \Phi^d_i}{\partial w^d_i(n)} = \sum_{q \in Q^d_i} \sum_{(j,k) \in q} p^d_{jk}(w^d(n)) = \sum_{q \in Q^d_{ii}} p_q(w^d(n)) = P^d_{ii}(w^d(n)). \]

By Proposition 3, total cycle probability for node \( i \), \( P^d_{ii} \), is componentwise decreasing in \( w^d_j \), \( \forall j \in N_d \). Therefore, the maximum \( P^d_{ii} \) is attained at the minimum expected costs, which is the solution of the system, i.e., at \( w^d \):

\[ \frac{\partial \Phi^d_i}{\partial w^d_i(n)} \leq \max_{w^d \in \Omega^d} \left\{ \frac{\partial \Phi^d_i}{\partial w^d_i(n)} \right\} = P^d_{ii}(w^d) = P^d_{ii}. \]

Then, \( P^d_c = \max_{i \in N_d} P^d_{ii} \) is a Lipschitz constant for the mapping \( \Phi^d \). Furthermore, \( P^d_c < 1 \) implies that \( \Phi^d \) is a contraction mapping, which is equivalent to the third statement of Theorem 1 that guarantees existence of the solution. Finally, Banach’s fixed point theorem implies that:

\[ \eta (w^d, w^d(n)) \leq \frac{(P^d_c)^n}{1 - P^d_c} \eta (w^{d(0)}, w^{d(1)}) \]

\[ \leq \frac{(P^d_c)^n}{1 - P^d_c} \eta (s^d, w^d) \]

\[ \max_{i \in N_d} \left\{ w^d_i(n) - w^d_i \right\} \leq \frac{(P^d_c)^n}{1 - P^d_c} \max_{i \in N_d} \left\{ s^d_i - w^d_i \right\}. \]

This completes the proof.

**Corollary 1.** If the traffic network is acyclic, Algorithm 1 has a quadratic convergence rate.

**Proof.** Since the network is acyclic, we have \( P^d_c = 0 \). Then, Proposition 4 directly implies quadratic convergence of the fixed point iterations. 

Corollary 1 implies quadratic convergence for acyclic networks. Actually, the solution can be obtained with a single pass if the expected costs are evaluated in the increasing order of the node labels defined in this section. However, traffic networks are usually cyclic, and Proposition 2 states that the algorithm, in general, has a linear convergence rate. This indicates the importance of the size of the search regions \( s^d_i - w^d_i \), especially when the choice probability of cyclic routes are considerably high. On the other hand,
Table 3: Nonincreasing expected cost series for $\theta = 2$.

<table>
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<tr>
<th>Iteration</th>
<th>$w_1^4$</th>
<th>$w_2^4$</th>
<th>$w_3^4$</th>
<th>$w_4^4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>6.0000</td>
<td>4.0000</td>
<td>4.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>1</td>
<td>5.9909</td>
<td>3.9365</td>
<td>3.9365</td>
<td>0.0000</td>
</tr>
<tr>
<td>2</td>
<td>5.9273</td>
<td>3.9275</td>
<td>3.9285</td>
<td>0.0000</td>
</tr>
<tr>
<td>3</td>
<td>5.9184</td>
<td>3.9273</td>
<td>3.9273</td>
<td>0.0000</td>
</tr>
<tr>
<td>4</td>
<td>5.9182</td>
<td>3.9273</td>
<td>3.9273</td>
<td>0.0000</td>
</tr>
<tr>
<td>5</td>
<td>5.9182</td>
<td>3.9273</td>
<td>3.9273</td>
<td>0.0000</td>
</tr>
</tbody>
</table>

the convergence rate approaches quadratic as $P_c^d \to 0$. Bounding the expected costs below improves both the linear convergence by narrowing down the search region $\Omega^d$, and the convergence rate by bounding $P_c^d$ above due to the monotonic relation between expected costs and cyclic route probabilities (Proposition 3). This concludes that the complexity of the Markovian SNL system depends on the distribution parameters. In the following subsection, we demonstrate on a numerical example this dependency, as well as the existence of the Markovian traffic assignment and convergence rate of the algorithm.

4.1 Numerical Example

Recall the four-node network in Figure 1 with a single destination $d = 4$. Note that there are infinitely many routes in this network due to the presence of the cycle between nodes 2 and 3.

Consider Case 1 given in Table 1 which corresponds to the Recursive Logit (RL) solution where $\theta_i^4 = 2, \forall i \in N_4$. The expected costs converge in five iterations with a precision of $10^{-4}$ as presented in Table 3. Small cycle probability of $P_c^4 = p_{23}^4 \times p_{32}^4 = 0.02$ indicates the efficiency of the solution approach. Case 2 in Table 1 corresponds to the RL solution where $\theta_i^4 = 1, \forall i \in N_4$. The algorithm in this case converges in seven iterations. The expected costs reduce to $w_1^4 = 5.41$ and $w_2^4 = w_3^4 = 3.54$; and the choice probability of the cycle increases to $P_c^4 = 0.14$. Expected costs and cycle choice probability continue to change monotonically in this manner as the dispersion parameter decreases.

Next, consider Case 3 in Table 1 which is the solution to the RL model with $\theta_i^4 = 0.1, \forall i \in N_4$. The algorithm converges in 59 iterations as presented in Table 4. This case illustrates that there might exist a solution to the system (4)-(9) with negative expected costs ($w_1^4 = -23.5030, w_2^4 = w_3^4 = -19.5216$). Choice probability of the cycle reaches around 0.82, indicating
Table 4: Nonincreasing expected cost series for $\theta = 0.1$.

<table>
<thead>
<tr>
<th>Iteration</th>
<th>$w_1^4$</th>
<th>$w_2^4$</th>
<th>$w_3^4$</th>
<th>$w_4^4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>6.0000</td>
<td>4.0000</td>
<td>4.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>1</td>
<td>0.0186</td>
<td>-2.4440</td>
<td>-2.4440</td>
<td>0.0000</td>
</tr>
<tr>
<td>2</td>
<td>-8.1942</td>
<td>-8.4254</td>
<td>-6.0194</td>
<td>0.0000</td>
</tr>
<tr>
<td>3</td>
<td>-13.2419</td>
<td>-11.5630</td>
<td>-10.1942</td>
<td>0.0000</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>57</td>
<td>-23.5029</td>
<td>-19.5216</td>
<td>-19.5216</td>
<td>0.0000</td>
</tr>
<tr>
<td>58</td>
<td>-23.5030</td>
<td>-19.5216</td>
<td>-19.5216</td>
<td>0.0000</td>
</tr>
<tr>
<td>59</td>
<td>-23.5030</td>
<td>-19.5216</td>
<td>-19.5216</td>
<td>0.0000</td>
</tr>
</tbody>
</table>

the slow convergence rate by Proposition 4. With an OD demand of 100, $h_1^4 = 100$, the flows on the links of the cycle, (2, 3) and (3, 2), are around 470. Therefore, link choice probabilities and flows become unintuitive for this setting of the distribution parameters.

Finally, we test a more extreme case when $\theta_i^4 = 0.001, \forall i \in N_4$. In this case, the algorithm converges only after 7569 iterations. The expected costs are $w_1^4 = -7594.40$ and $w_2^4 = w_3^4 = -6904.26$; and the flows on the links of the cycle are around 7500. The transition matrix $P_d$ and its representation with a working precision of two, $P_d^{(2)}$, are given below:

$$P_d = \begin{bmatrix}
1 & 2 & 3 & 4 \\
1 & 0 & 0.5005 & 0.4995 & 0 \\
2 & 0 & 0 & 0.9990 & 0.0010 \\
3 & 0 & 0.9990 & 0 & 0.0010 \\
4 & 0 & 0 & 0 & 1.000 \\
\end{bmatrix}, \quad P_d^{(2)} = \begin{bmatrix}
1 & 2 & 3 & 4 \\
1 & 0 & 0.50 & 0.50 & 0 \\
2 & 0 & 0 & 1.00 & 0 \\
3 & 0 & 1.00 & 0 & 0 \\
4 & 0 & 0 & 0 & 1.00 \\
\end{bmatrix}.$$

Assuming the machine runs with a precision of two decimals, the cycle including nodes 2 and 3 form a recurrent class, and $I - Q_d^{(2)}$ becomes singular. By Proposition 3, cycle choice probability $p_{23}^4 \times p_{32}^4$ increases monotonically as the dispersion parameter decreases. Therefore, for any level of precision, $m$, there exists a dispersion parameter for which $I - Q_d^{(m)}$ is singular. This illustrates that even for the logit case where existence is theoretically guaranteed, the solution cannot be obtained for certain distribution parameters.

In order to illustrate the theoretical divergence of the system, we use the Markovian Marginal Distribution Model proposed by Alpasanoglu et al. (2017). M-MDM allows to use different families of marginals distributions and nonzero error means. We assume that the marginals of the random error terms are exponential with a location parameter of zero and a scale
Table 5: Nonincreasing expected cost series for M-MDM with a negative cost cycle.

<table>
<thead>
<tr>
<th>Iteration</th>
<th>$w_1^4$</th>
<th>$w_2^4$</th>
<th>$w_3^4$</th>
<th>$w_4^4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>6.0000</td>
<td>4.0000</td>
<td>4.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>1</td>
<td>3.3735</td>
<td>1.0518</td>
<td>1.0518</td>
<td>0.0000</td>
</tr>
<tr>
<td>2</td>
<td>-0.1628</td>
<td>-1.8963</td>
<td>-0.5888</td>
<td>0.0000</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>11</td>
<td>-19.2416</td>
<td>-20.2935</td>
<td>-19.2935</td>
<td>0.0000</td>
</tr>
<tr>
<td>12</td>
<td>-21.2417</td>
<td>-22.2935</td>
<td>-21.2935</td>
<td>0.0000</td>
</tr>
<tr>
<td>13</td>
<td>-23.2417</td>
<td>-24.2935</td>
<td>-23.2935</td>
<td>0.0000</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>498</td>
<td>-993.2417</td>
<td>-994.2936</td>
<td>-993.2936</td>
<td>0.0000</td>
</tr>
<tr>
<td>499</td>
<td>-995.2417</td>
<td>-996.2936</td>
<td>-995.2936</td>
<td>0.0000</td>
</tr>
<tr>
<td>500</td>
<td>-997.2417</td>
<td>-998.2936</td>
<td>-997.2936</td>
<td>0.0000</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
</tbody>
</table>

Expected perceived link costs of the cycle become $s_{23}^4 = s_{32}^4 = 1 - 2 = -1$, and hence the cycle becomes a negative cost cycle with $s_{23}^4 + s_{32}^4 = -2$. Table 5 displays the first 500 iterations with the same initial solution. After 12 iterations, the algorithm reaches a stable behavior where the expected costs reduce by two, which is the cost of the cycle, at every iteration. This behavior continues indefinitely, and the expected costs diverge to negative infinity as stated in Proposition 1.

5 Restricting the Parameter Space

Our results so far indicate that satisfying the existence condition is not sufficient to achieve an efficient convergence rate, nor to obtain a realistic traffic assignment. Proposition 3 shows the relation between expected costs and traffic flows on the links of cycles. This allows us to focus on the subsystem (4)-(5).

As the expected cost at the destination is zero by the boundary condition, it is intuitive to expect nonnegative expected cost from transient nodes to the destination, $w_i^d \geq 0, \forall i \in N_d, d \in D$. Furthermore, from a myopic perspective, it is intuitive to expect that the expected instantaneous disutility of choosing a link at a particular node is nonnegative. Let $\phi_i^d(t, \pi)$ be the expected instantaneous disutility at node $i$ where $\pi$ represents the
vector of distribution parameters of the joint distribution of random utilities $(\tilde{\epsilon}_{ij})_{j \in N_d^+ (i)}$. Then for a given link cost vector $t$, the above statement is equivalent to the following:

$$
\phi_i^d (t, \pi) = E_{\pi} \left[ \min_{j \in N_d^+ (i)} \left\{ t_{ij} - \tilde{\epsilon}_{ij}^d \right\} \right] \geq 0, \quad \forall i \in N_d, d \in D,
$$

where the expectation is calculated with respect to the parameter vector $\pi$.

Let $\Pi_i^d$ be the space of parameters of the joint distribution of random utilities $(\tilde{\epsilon}_{ij})_{j \in N_d^+ (i)}$ for node $i \in N_d$ and destination $d$. We define the restricted parameter space $\pi_i^d \subseteq \Pi_i^d$ as follows:

$$
\pi_i^d = \left\{ \pi \in \Pi_i^d : \phi_i^d (t, \pi) \geq 0 \right\}, \quad \forall i \in N_d, d \in D. \tag{17}
$$

This basic idea leads to sufficient conditions on the parameter space to guarantee nonnegative expected costs for the Markovian traffic assignment.

**Proposition 5.** Assume that the distribution parameters of the Markovian choice model lie in the restricted parameter space $\pi_i^d$ defined in (17). Then, the solution of the Markovian SNL equations (4)-(9) satisfies nonnegative expected costs:

$$
w_i^d \geq 0, \quad \forall i \in N_d, d \in D.
$$

**Proof.** Let $\pi_i^{d^*}$ be the vector of distribution parameters on the boundary of $\pi_i^d$ satisfying zero instantaneous costs:

$$
\phi_i^d (t, \pi_i^{d^*}) = E_{\pi_i^{d^*}} \left[ \min_{j \in N_d^+ (i)} \left\{ t_{ij} - \tilde{\epsilon}_{ij}^d \right\} \right] = 0, \quad \forall i \in N_d, d \in D.
$$

In this case, $w_i^d = 0, \forall i \in N_d$ is a trivial solution to the system (4)-(5). Since the system has at most one solution by Theorem 1, this must be the unique solution. Since $\phi_i^d (t, w^d)$ is monotonically increasing in $w_j^d, \forall j \in N_d$, we have $w_i^d \geq 0, \forall i \in N_d$ for any parameter setting within the restricted parameter space $\pi_i^d, \forall i \in N_d, d \in D$. This completes the proof. \hfill \Box

The results presented in previous sections indicate the importance of keeping the expected costs nonnegative. Baillon and Cominetti (2008) also include the constraint $w_i^d \geq 0, \forall i \in N_d, d \in D$ in their equivalent MTE formulation. It is important to note the difference and the relation of this formulation with our approach. Consider Case 3 presented in Table 1 which
corresponds to the Recursive Logit solution with negative expected costs as presented in Table 4. For the sake of brevity, we ignore the effect of congestion. The MTE solution for this RL setting would be \( w_d^i = 0, \forall i \in N_d \) due to the nonnegativity constraint. However, choice probabilities and link flows obtained using zero expected costs do not represent the choice behavior with a dispersion parameter of \( \theta = 0.1 \). This is more clear when we consider the fact that the MTE solution would be the same for any dispersion parameter less than 0.1. Proposition 5, on the other hand, provides an approach to achieve nonnegative costs by restricting the parameter space. For Case 3, it prevents setting the dispersion parameter to 0.1 to eliminate the solution with negative expected costs. In other words, if the choice model is estimated within the restricted parameter space, the nonnegativity constraint of the MTE formulation is automatically satisfied.

Satisfaction function (10) is usually monotonic in distribution parameters. This allows to efficiently obtain the restricted parameter space defined in 17. In the following corollary, we present the application of Proposition 5 to the Recursive Logit model.

**Corollary 2.** Consider the Markovian choice model where the error terms are iid Gumbel random variables with dispersion parameters \( \theta_d^i, \forall i \in N_d \), which leads to choice probability and expected minimum cost expressions given in (13) and (14), respectively [Akamatsu, 1996; Fosgerau et al., 2013; Mai et al., 2015]. Let \( N'_d \subseteq N_d \) be the set of nodes which have at least two emanating links in \( G_d \). Define the lower bound \( \underline{\theta}_d^i > 0 \) as the unique value satisfying the following:

\[
\sum_{j \in N_d^i(i)} \exp \left( -\underline{\theta}_d^i t_{ij} \right) = 1, \quad \forall i \in N'_d, d \in D. \tag{18}
\]

If the dispersion parameters satisfy the following inequality:

\[
\theta_d^i \geq \underline{\theta}_d^i, \quad \forall i \in N'_d, d \in D,
\]

then the solution of the Markovian SNL equations (4) - (9) satisfies nonnegative expected costs, \( w_d^i \geq 0, \forall i \in N_d, d \in D \).

**Proof.** Note that if node \( i \) has a single emanating link \((i, j)\), then we deterministically have \( p_d^i = 1 \) and \( w_d^i = w_d^j + t_{ij} \), which satisfies \( w_d^i > 0 \) provided that \( w_d^j \geq 0 \). Therefore, we focus on the nodes in \( N'_d \). The choice model has a single parameter and the condition \( \pi \in \pi_d^i \) is equivalent to \( \theta_d^i \geq \underline{\theta}_d^i \). Therefore, \( \theta_d^i \geq \underline{\theta}_d^i \) is sufficient to guarantee \( w_d^i \geq 0, \forall i \in N_d \) by Proposition 5. This completes the proof.
Lower bounds $\theta^d$ can be obtained by solving (18) with simple root finding methods. Therefore, Corollary 2 provides practical and linear constraints on the parameter space to guarantee nonnegative costs for the logit setting, which not only helps providing realistic traffic flows but also deals with the computational inefficiencies in solving the Markovian system.

For a second application of the approach, we use M-MDM that is recently proposed in [Ahipašaoğlu et al. (2017)]. The model provides a distributionally robust Markovian choice model building on the models proposed in Nataraajan et al. (2009) and Mishra et al. (2014). Here, it is sufficient to note that M-MDM uses the marginal distribution information, while expected costs and choice probabilities are calculated using an extremal joint distribution. Satisfaction function in (10) and choice probability expression in (6) for M-MDM take the following forms, respectively:

\[
\varphi^d_i(t, w^d) = \max_{\lambda^d} \left\{ -\lambda - \sum_{j \in N^d(i)} \int_{\lambda^d + t_{ij} + w^d_j}^{\infty} \left[ 1 - F_{ij}^d(\omega) \right] d\omega \right\},
\forall i \in N_d, d \in D, (19)
\]

\[
\rho^d_{ij} = 1 - F_{ij}^d(\lambda^d_i + t_{ij} + w^d_j), \quad \forall (i, j) \in A_d, d \in D, (20)
\]

where $F_{ij}^d(.)$ is the cumulative distribution function (cdf) of the marginal distribution of the error term $\tilde{e}_{ij}^d$; and $\lambda^d_i$ is the unique maximizer of the problem in (19). Provided that the cdf is monotone in marginal distribution parameters, simple line search methods integrated with the convex optimization problem in (19) can be used to restrict the parameter space by Proposition 5. Note that most distribution families satisfy this property. The following corollary provides the conditions for the special case with exponential marginals where M-MDM has a closed form expression for expected costs.

**Corollary 3.** Consider the Markovian Marginal Distribution Model where the marginal distribution of the error term $\tilde{e}_{ij}^d$ is exponential with location parameter $\alpha^d_{ij}$ and scale parameter $\beta^d_{ij}$:

\[
F_{ij}^d(\omega) = 1 - \exp \left( \frac{\alpha^d_{ij} - \omega}{\beta^d_i} \right), \quad \forall \omega \geq \alpha^d_{ij}, (i, j) \in A_d, d \in D.
\]
If the marginal distribution parameters satisfy the following:

\[ \sum_{j \in N_i^d(t)} \exp \left( \frac{\alpha_{ij}^d - t_{ij}}{\beta_i^d} \right) \leq \exp(-1), \quad \forall i \in N_d, d \in D, \quad (21) \]

then the solution to the Markovian SNL system satisfies nonnegative expected costs, \( w_i^d \geq 0, \forall i \in N_d, d \in D. \)

The proof is provided in Appendix A.2. The condition in (21) provides a lower bound on the scale parameter given the location parameters, or upper bounds on location parameters given the scale parameter; and hence, defines the restricted parameter space.

6 Markovian Traffic Equilibrium

Stochastic network loading ignores the effect of congestion and loads the network according to the underlying discrete choice model at a given link cost vector. In congested networks, cost of a link increases as more users follow routes using that link. The relation is expressed with link cost functions, \( \tau_{ij}(\cdot). \) A commonly used link cost function is the Bureau of Public Roads (BPR) function:

\[ \tau_{ij}(f_{ij}) = t_{ij}^0 \left[ 1 + \alpha_{ij} \left( \frac{f_{ij}}{q_{ij}} \right)^\gamma_{ij} \right], \quad \forall (i,j) \in A, \]

where \( t_{ij}^0 = \tau_{ij}(0) \) is the free-flow travel cost, \( q_{ij} \) is the capacity, and \( \alpha_{ij} \) and \( \gamma_{ij} \) are constants. Therefore, \( f_{ij}/q_{ij} \) serves as a measure of congestion. Link cost functions are usually assumed to have the following property:

P4 Link cost functions are strictly increasing in link flow and \( \tau_{ij}(f) \geq t_{ij}^0 > 0, \forall (i,j) \in A, f \geq 0. \)

Markovian traffic equilibrium can be defined by introducing the congestion effect to the Markovian SNL equations. For completeness, we give the
entire system of equations that characterize MTE flows below:

\[
\begin{align*}
  w_i^d &= \varphi_i^d \left( t, w^d \right), & \forall i \in N_d, d \in D, \\
  w_d^d &= 0, & \forall d \in D, \\
  p_{ij}^d &= \Pr \left( j \in \arg\min_{k \in N^+_d(i)} \left\{ t_{ik} + w_k^d - \tilde{c}_{ik}^d \right\} \right), & \forall (i, j) \in A_d, d \in D, \\
  n^d &= [I - Q^d]^{-1} h^d, & \forall d \in D, \\
  g_{ij}^d &= n_i^d p_{ij}^d, & \forall (i, j) \in A_d, d \in D, \\
  f_{ij} &= \sum_{d \in D : (i, j) \in A_d} g_{ij}^d, & \forall (i, j) \in A, \\
  t_{ij} &= \tau_{ij}(f_{ij}), & \forall (i, j) \in A.
\end{align*}
\]

Results presented in this section are straightforward extensions of our findings for the Markovian choice model to the congested network. The underlying idea is that if the Markovian choice model has a solution for the uncongested network, the equilibrium also exists. Similarly, if the convergence of Algorithm 1 is efficient with free-flow travel costs, it will also be efficient for the congested network.

**Theorem 2.** If there exists no negative cost cycle in \( G_d \) with respect to expected perceived link costs \( s^0_{ij} := t^0_{ij} - E[\tilde{c}_{ij}^d] \) for any \( d \in D \), then there exists a unique solution to the MTE system of equations (22)-(28).

**Proposition 6.** Assume that the distribution parameters of the Markovian choice model lie in the restricted parameter spaces \( \pi_i^d \) defined as:

\[
\pi_i^d = \left\{ \pi \in \Pi_i^d : \varphi_i^d (t^0, \pi) \geq 0 \right\}, \quad \forall i \in N_d, d \in D.
\]

Then, the solution of the MTE equations (22)-(28) satisfies nonnegative expected minimum perceived costs:

\[
w_i^d \geq 0, \quad \forall i \in N_d, d \in D.
\]

Note that \( t_{ij} = \tau_{ij}(f_{ij}) \geq t^0_{ij} \), and hence, \( s_{ij}^d \geq s^0_{ij}, \forall (i, j) \in A_d \) for any flow vector \( f \). Then, the proofs of Theorem 2 and Proposition 6 follow directly from Theorem 1 and Proposition 5 due to the monotonicity of the satisfaction function \( \varphi_i^d \) in \( w_j^d, \forall j \in N_d \) and in \( t_{ij}, \forall (i, j) \in A_d \). The condition in Theorem 2 is equivalent to the necessary existence condition in Theorem
for free-flow travel costs, i.e., for $t = t^0$. It is intuitive to expect that the Markovian choice model has a solution for the uncongested network, which is a sufficient condition for the existence of the equilibrium flows for the congested case. Similarly, nonnegativity of expected costs in the uncongested network is a sufficient condition for nonnegative expected costs at the traffic equilibrium.

7 Computational Study

In this section, we experiment on our results on the four-node network in Figure 1 and the network of Winnipeg. The topology, link characteristics, and OD demands of Winnipeg network are obtained from Bar-Gera’s Traffic Assignment Test Problem website. Our experiments show that when the error distribution parameters are estimated within the restricted parameter space, Markovian traffic assignment and Markovian traffic equilibrium can be obtained efficiently. Furthermore, convergence performance does not vary significantly within the restricted region. On the other hand, we observe that computational burden is very sensitive to the distribution parameters outside the proposed region. In particular, solution time increases drastically, and the traffic assignment becomes extremely unintuitive.

7.1 Convergence Behavior

Let RL($\theta$) represent the Recursive Logit model where the dispersion parameter for all nodes is $\theta$, i.e., $\theta_i^d = \theta, \forall i \in N_d, d \in D$. In this section, we test the convergence behavior of Algorithm 1 with respect to the stopping condition. Recall that we use the threshold $\xi$ in following stopping condition:

$$\max_{i \in N_d} \left\{ w_i^{d(n-1)} - w_i^{d(n)} \right\} < \xi,$$

where $n$ is the iteration number. In order to test the convergence rate of the algorithm, we use the Winnipeg network. We plot the number of iterations of the algorithm for RL(1) against the threshold $\xi$ in Figure 2. We observe that the algorithm shows a linear rate of convergence when the threshold is reasonably large (greater than 0.03%).

\footnote{All experiments are carried out with the open-source software seSue published in \url{http://people.sutd.edu.sg/~ugur_arikan/seSue/}.}

\footnote{See \url{https://github.com/bstabler/TransportationNetworks}.}
Figure 2: Convergence behavior of Algorithm 1 with respect to error threshold.

7.2 Experiments with Recursive Logit

In this section, we test our general theoretical results using the RL model. We use the small network illustrated in Figure 1. Markovian SNL problem is solved for dispersion parameter varying in the interval \((0, 2]\).

Figure 3 plots the expected costs at nodes 2 and 3 for \(\text{RL}(\theta)\) with respect to \(\theta\). Since \(\bar{s}_2^4 = \bar{s}_3^4 = 4\), the costs are bounded above by 4 regardless of the dispersion parameter due to Lemma 1. Restricted parameter space that guarantees nonnegative costs is defined as \(\theta \geq \theta_4^2 = \theta_3^4 = 0.3223\) by Corollary 2. This bound corresponds to the intercept in Figure 3. The costs vary within the reasonable region \([0, 4]\) for dispersion parameters within the restricted parameter space. When the value of the dispersion parameter drops below the threshold, we observe that the expected costs drop drastically. Therefore, restricted parameter space defined in Proposition 5 eliminates the region that leads to extremely unrealistic traffic assignment.

Figure 4 plots the number of iterations to solve the Markovian SNL system \((4)-(9)\) against the dispersion parameter \(\theta\). We observe fast convergence for \(\theta \geq 0.3223\), and number of iterations does not vary significantly in this restricted region. On the other hand, in the negative expected cost region, convergence might be extremely slow, requiring up to 700 iterations for this small example. One reason for this poor performance is due to the increase in the size of the search region. Recall that Algorithm 1 creates a nonincreasing series of the expected costs starting at \(\bar{s}_i^d\) converging to \(w_i^d\). It can
be observed in Figure 3 that the region \([\omega_1, \omega_2]\) is bounded in the restricted parameter space, while it can be extremely wide in the negative expected cost region. Another result of Proposition 4 is that the convergence rate improves as the maximum cycle choice probability, \(P_{c_1}^d\), decreases. This relation is observed in Figure 4. Finally, we know that the cycle choice probability \(p_{23} \times p_{32}^d\) never reaches one, and this guarantees that the solution to the RL model exists unconditionally by Theorem 1. On the other hand, we observe in the figure that it reaches the neighborhood of one for very small values of the dispersion parameter, resulting in nonsingularity of the matrix \(I - Q_d\) due to machine precision. On the other hand, \(P_{c_1}^d\) is sufficiently less than one in the restricted parameter space preventing this problem.

### 7.3 Experiments with Markovian Marginal Distribution Model

In this section, we repeat the experiments carried out for the RL model with the Markovian Marginal Distribution Model. In particular, we test the results using M-MDM with exponential marginals. For each value of the scale
parameter, we obtain an upper bound on the location parameter using inequality (21) of Corollary 3 to define the restricted parameter space. Figure 5 presents the contour plot of expected costs of nodes 2 and 3 solving the Markovian system (4)-(5). The contour line with a value of zero corresponds to the boundary that restricts the parameter space. We again observe that the expected costs decrease on the right-hand-side of this boundary, and the assignment rapidly reaches solutions with extremely unrealistic costs. The effect of this on the convergence performance is illustrated in Figure 6. Number of iterations required to solve the system is less than 10 for any setting within the restricted parameter space, which is the left-hand-side of the dashed contour. However, when we leave this region, number of iterations increases rapidly and exceeds a hundred iterations.

7.4 Large Network

In order to test our results on a larger instance, we use the Winnipeg network, which is composed of 1040 nodes, 2836 links and 4344 OD pairs. The network has 138 distinct destinations, illustrating a case where |W| ≫ |D|.

We start our analysis with the Markovian choice model for the uncongested network. We solve the Markovian SNL equations (4)-(9) for each destination. In order to test the existence of the Markovian traffic assignment and convergence performance, we use the Recursive Logit model under the following experimental setup. We use equation (18) in Corollary 2 to calculate θ^d_i, ∀i ∈ N_d, d ∈ D. Recall that the condition θ^d_i ≥ θ^d_i, ∀i ∈ N_d, d ∈ D guarantees that expected costs solving the subsystem (4)-(5) are nonneg-
Figure 5: Effect of scale and location parameters of M-MDM on expected costs.

Figure 6: Effect of scale and location parameters of M-MDM on cycle probability and convergence rate of Algorithm 1 in the RL model.
We define the positive multiplier $\kappa$ and dispersion parameters as $\theta_i^d = \kappa \theta_i^d, \forall i \in N_d, d \in D$. Then by Proposition 5, the solution will have negative expected costs for $\kappa < 1$ and nonnegative expected costs for $\kappa \geq 1$. We solve the Markovian system for $\kappa \in \{0.90, 0.91, \ldots, 2.00\}$. Figure 7 plots number of iterations for the subsystem (4)-(5) and total solution time of Algorithm 1 for different values of $\kappa$. The figure plots aggregate values for all 138 destinations. The missing points for $\kappa \leq 0.95$ indicate that the Markovian assignment could not be obtained. This is due to the machine precision problem discussed before, where the choice probabilities of some cycle is very close to one. Furthermore, there is a drastic increase in the solution time and number of iterations when $\kappa$ reduces from 1.00 to 0.96. This again indicates the inefficiency of the solution approach in the negative expected cost region. On the other hand, for $\kappa \geq 1.00$, the solution performance improves. We observe that the convergence rate is not very sensitive to $\kappa$, or equivalently to distribution parameters. These experiments show that Proposition 5 provides a practical approach to restrict the parameter space to guarantee efficient convergence of the solution algorithm.

Next, we investigate the computational burden of solving the two system of equations for the Markovian SNL mechanism. The first one is the nonlinear system of equations defined by equations (4)-(5) (lines 6-12 in Algorithm 1); and the second one is the linear system defined in (7) (line 16 in Algorithm 1). Figure 8 represents the overall distribution of the solution time.
to these systems and to the remaining assignment steps. The figure points out the criticality of the subsystem \([4)-(5]\) in terms of computational performance. The average out degree of the vertices in the Winnipeg network is around 2.73. Therefore, the fraction of nonzero entries of the matrix \(Q_d\) is around \(2.73/|N_d| = 2.73/1039 = 0.002628\), and the sparsity helps reduce the time required to solve the linear system (7).

Finally, we test our results for the Markovian traffic equilibrium taking congestion into account. We integrate Algorithm 1 in the Method of Successive Averages algorithm (Sheffi, 1985) as the SNL mechanism. Figure 9 plots the number of MSA iterations for the equilibrium problem (22)-(28) and solution time of the MSA algorithm for different values of \(\kappa\). Although number of MSA iterations has a decreasing trend as \(\kappa\) increases, the reduction is not as significant as for the choice model. On the other hand, we observe a similar behavior in solution times, since solution time of the subproblems (4)-(9) increases as \(\kappa\) decreases. Solution time reaches up to four hours when \(\kappa = 0.96\) and the algorithm fails to obtain the equilibrium flows for \(\kappa \leq 0.95\). On the other hand, unlike the drastic variation in the negative cost region, solution time becomes less sensitive to the distribution parameters for \(\kappa \geq 1\). Average solution time for \(\kappa \in [1.25, 2.00]\) is around 18.69 minutes. These experiments indicate that the solution performance of the MSA algorithm can be considerably inefficient. On the other hand, working within the restricted parameter space guarantees efficient convergence with predictable solution times.

8 Conclusion

Akamatsu (1996) proposed the idea of using a Markovian decision process to model the route choice, and Baillon and Cominetti (2008) provided a framework for the Markovian traffic equilibrium models under general random utility maximization. The idea has important advantages over the route
based models, such as relaxing the predetermined route choice set assumption and alleviating the need for route generation. However, it has attracted lesser attention due to the additional complexity of the dynamic choice process. There exist serious issues noted by several authors, which are unlikely to be observed in traditional route based models. This study addresses these issues by theoretically explaining the underlying reasons and proposing an approach to improve the practicality of Markovian models.

First issue is that the Markovian traffic equilibrium might not exist. We propose the necessary and sufficient existence condition for the Markovian choice model and extend the condition to the congested network. Fortunately, the condition depends only on the means of the error terms and the network topology. This allows to easily test the existence of the equilibrium. Secondly, solving the nonlinear system of equations of the Markovian choice model might be extremely inefficient. Lastly, traffic assignment under the Markovian choice model can be unacceptable. In particular, the solution might assign the majority of the users to cyclic routes, and the link flows increase unrealistically. We prove that these two issues are related. In particular, we show that if the cyclic flows increase, complexity of the system also increases. On the other hand, the nonlinear system can efficiently be solved when the cyclic flows are kept within reasonable limits. These findings lead to the idea of eliminating the region where the Markovian model performs poor both behaviorally and computationally. We propose a practical
approach to restrict the parameter space so that the Markovian assignment within the restricted space guarantees bounded cyclic flows and efficient convergence behavior. We test and validate our approach with experiments on small and large networks.

Finally, we propose a Markovian stochastic network loading algorithm for the general case, which performs fixed-point iterations with a proposed initial solution, and present its convergence analysis. For our experiments on congested networks, we integrate the algorithm as an inner procedure to the Method of Successive Averages algorithm. It is known that the MSA algorithm might have slow convergence. Although the solution algorithm for the equilibrium problem is not considered in this paper, an interesting topic for future research is to develop an efficient equilibrium algorithm that exploits the characteristics of the Markovian choice models.

A Appendices

A.1 Node Labeling

In this section, we show that the node labeling used in the proof of Proposition 2 exists and provide a simple algorithm to assign the node labels. Recall the definitions of \([l], T_l, D_l, C_l\) given in Section 3. Let \(i \not\rightarrow j\) denote that there exists no path from node \(i\) to node \(j\). We define parameter \(\phi_{ij}, \forall i, j \in N_d\), as follows:

\[
\phi_{ij} = \begin{cases} 
1, & \text{if } (i \rightarrow j \text{ and } j \rightarrow i) \text{ or } (i \not\rightarrow j \text{ and } j \not\rightarrow i) \text{ or } (i \not\rightarrow j \text{ and } j \rightarrow i), \\
0, & \text{if } (i \not\rightarrow j \text{ and } j \not\rightarrow i).
\end{cases}
\]

In the following lemma, we show that the node labeling with defined properties can be generated with Algorithm 2.

**Lemma 3.** The labels assigned by Algorithm 2 satisfy the following properties:

\[
[l] = d, \\
D_l \neq \emptyset, \quad \forall l \in \{2, 3, \ldots, |N_d| + 1\}, \\
j \mapsto [l], \quad \forall j \in C_l, l \in \{2, 3, \ldots, |N_d| + 1\}.
\]

**Proof.** First condition is satisfied by the initialization condition. Consider step \(l\) in the algorithm for any \(l \in \{2, 3, \ldots, |N_d| + 1\}\). The conditions that there exists a path from each node in \(N_d\) to destination \(d\) and \(T_l \supseteq \{d\}\) for
Algorithm 2 Labeling Algorithm

Input: $G_d = (N_d, A_d)$

1: procedure LA($G_d$)
2: \[1\] = $d$, $S_1 = N_d$, $T_1 = \{d\}$.
3: for $l = 2$ to $|N_d| + 1$ do
4: \[I\] = $\{i \in S_{l-1} : N_d^+(i) \cap T_{l-1} \neq \emptyset\}$.
5: $I' = \begin{cases} I, & \text{if } |I| = 1, \\
\{i \in I : \phi_{ij} = 1, \forall j \in I \setminus \{i\}\}, & \text{otherwise.}
\end{cases}$
6: arbitrarily select $i^*$ from $I'$.
7: \[l\] = $i^*$, $S_l = S_{l-1} \setminus \{i^*\}$, $T_l = T_{l-1} \cup \{i^*\}$.
8: end for
9: end procedure

Output: \[l\], $l \in \{1, 2, \ldots, |N_d| + 1\}$.

any $l$, the subset $I$ is nonempty. Similarly, $I' \subseteq I$ is nonempty by definition in line 5. Therefore, the algorithm converges to $T_{|N_d|+1} = N_d \cup \{d\}$. Since the node $i^*$ is chosen among the nodes in the subset $I' \subseteq I$, the second condition is satisfied by the definition of the subset $I$ in line 4. For any node $j \in N_d^+(\[l\])$, we have \[l\] $\rightarrow$ $j$ since $(\[l\], j) \in A_d$. Now consider the case where $j \not\rightarrow \[l\]$ for some $j \in N_d^+(\[l\])$. Due to the definition of the parameter $\phi_{ij}$ and subset $I'$, node $j$ must be labeled prior to node $\[l\]$; i.e., $j \in T_{l-1}$. Therefore, if $j \in N_d^+(\[l\]) \setminus T_{l-1}$, we must have $j \rightarrow \[l\]$. This guarantees that the third condition is satisfied, and completes the proof.

A.2 Proof of Corollary 3

Proof. Consider the special case of the M-MDM defined in Corollary 3. It can be shown that the variable maximizing the problem in (19) for the myopic case, $w_d^j = 0$, $\forall j \in N_d^+(i)$, has the following closed form expression:

$$
\lambda_d^i = \beta_d^i \ln \left( \sum_{j \in N_d^+(i)} \exp \left( \frac{\alpha_{ij}^d - t_{ij}}{\beta_d^i} \right) \right), \quad \forall i \in N_d, j \in N_d^+(i), d \in D.
$$

Substituting these dual variables in (19), we obtain the following closed form expression for the satisfaction function:

$$
\varphi^d_i(t) = -\beta_d^i \ln \left( \sum_{j \in N_d^+(i)} \exp \left( \frac{\alpha_{ij}^d - t_{ij}}{\beta_d^i} \right) \right) - \beta_d^i, \quad \forall i \in N_d, d \in D.
$$
Then, the condition $\varphi^d_i(t) \geq 0$ is equivalent to:

$$\sum_{j \in N^+_d(i)} \exp \left( \frac{\alpha_{ij}^d - t_{ij}}{\beta^d_i} \right) \leq \exp(-1), \quad \forall i \in N_d, d \in D.$$  

This completes the proof. \qed
References


