Pairwise independent correlation gap (Additional Details)

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0.1 Verifying the bound for regions $R_2 - R_5$ in the main paper

Proof. For ease of readability, we provide the proof in tabular format with entries $\overline{f}^{+}(\boldsymbol{x})/\underline{f}^{++}(\boldsymbol{x})$. Tables 1, 2, 3, 4 display the results for regions R_2, R_3, R_4, R_5 respectively for each of the chosen dual solutions and valid extremal submodular functions. For ease of verification of the 4/3 bound, $\overline{f}^{+}(\boldsymbol{x})$ has been re-written in the form of the objective of the primal linear program (as the expected function value over a joint distribution) for each region and chosen dual solution. The last column of each table shows the inequality (I_1) or (I_2) required to prove the bound along with the parameters (α, β) (shown in brackets) in terms of the marginal probabilities x_1, x_2, x_3. For regions R_2, R_3, R_4, the excluded extreme points of $\mathcal{E}\left(\mathcal{F}_2^1\right)$, $\mathcal{E}\left(\mathcal{F}_2^2\right)$, $\mathcal{E}\left(\mathcal{F}_2^3\right)$ are denoted with $N.A.$

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$\frac{\overline{f}^{+}(\boldsymbol{x})}{\underline{f}^{++}(\boldsymbol{x})}$	Extreme point	D_2 $\left[\overline{f}^{+}(\boldsymbol{x}) = \begin{array}{r} f(\{1\})x_1 + f(\{2\})(1-x_1-x_3) + f(\{3\}) \\ (1-x_1-x_2) + f(\{2,3\})(x_1+x_2+x_3-1) \end{array}\right]$	D_3 $\left[\overline{f}^{+}(\boldsymbol{x}) = \begin{array}{r} f(\{1\})(1-x_2-x_3) + f(\{2\})x_2 + f(\{3\}) \\ (1-x_1-x_2) + f(\{1,3\})(x_1+x_2+x_3-1) \end{array}\right]$	D_4 $\left[\overline{f}^{+}(\boldsymbol{x}) = \begin{array}{r} f(\{1\})(1-x_2-x_3) + f(\{2\})(1-x_1-x_3) \\ +f(\{3\})x_3 + f(\{1,2\})(x_1+x_2+x_3-1) \end{array}\right]$	Inequality used (α,β in brackets)
E_1	1 0 0 1 1 0	$\frac{x_1}{x_1} \leq \frac{4}{3}$	$\frac{(1-x_2-x_3)+(x_1+x_2+x_3-1)}{x_1} = \frac{x_1}{x_1} \leq \frac{4}{3}$	$\frac{(1-x_2-x_3)+(x_1+x_2+x_3-1)}{x_1} = \frac{x_1}{x_1} \leq \frac{4}{3}$	
E_2	0 1 0 1 0 1	$\frac{(1-x_1-x_3)+(x_1+x_2+x_3-1)}{x_2} = \frac{x_2}{x_2} \leq \frac{4}{3}$	$\frac{x_2}{x_2} \leq \frac{4}{3}$	$\frac{(1-x_1-x_3)+(x_1+x_2+x_3-1)}{x_2} = \frac{x_2}{x_2} \leq \frac{4}{3}$	
E_3	0 0 1 0 1 1	$\frac{(1-x_1-x_2)+(x_1+x_2+x_3-1)}{x_3} = \frac{x_3}{x_3} \leq \frac{4}{3}$	$\frac{(1-x_1-x_2)+(x_1+x_2+x_3-1)}{x_3} = \frac{x_3}{x_3} \leq \frac{4}{3}$	$\frac{x_3}{x_3} \leq \frac{4}{3}$	
E_4	1 1 0 1 1 1	$\frac{x_1+(1-x_1-x_3)+(x_1+x_2+x_3-1)}{x_1+x_2-x_1x_2} = \frac{x_1+x_2}{x_1+x_2-x_1x_2} \leq \frac{4}{3}$	$\frac{(1-x_2-x_3)+x_2+(x_1+x_2+x_3-1)}{x_1+x_2-x_1x_2} = \frac{x_1+x_2}{x_1+x_2-x_1x_2} \leq \frac{4}{3}$	*N.A.*	I_1 (x_1,x_2)
E_5	1 0 1 1 1 1	$\frac{x_1+(1-x_1-x_2)+(x_1+x_2+x_3-1)}{x_1+x_3-x_1x_3} = \frac{x_1+x_3}{x_1+x_3-x_1x_3} \leq \frac{4}{3}$	*N.A.*	$\frac{(1-x_2-x_3)+x_3+(x_1+x_2+x_3-1)}{x_1+x_3-x_1x_3} = \frac{x_1+x_3}{x_1+x_3-x_1x_3} \leq \frac{4}{3}$	I_1 (x_1,x_3)
E_6	0 1 1 1 1 1	*N.A.*	$\frac{x_2+(1-x_1-x_2)+(x_1+x_2+x_3-1)}{x_2+x_3-x_2x_3} = \frac{x_2+x_3}{x_2+x_3-x_2x_3} \leq \frac{4}{3}$	$\frac{(1-x_1-x_3)+x_3+(x_1+x_2+x_3-1)}{x_2+x_3-x_2x_3} = \frac{x_2+x_3}{x_2+x_3-x_2x_3} \leq \frac{4}{3}$	I_1 (x_2,x_3)
E_7	1 1 1 1 1 1	$\frac{1}{(x_1+x_2)+x_3-x_3(x_1+x_2)} \leq \frac{4}{3}$	$\frac{1}{(x_1+x_2)+x_3-x_3(x_1+x_2)} \leq \frac{4}{3}$	$\frac{1}{(x_1+x_2)+x_3-x_3(x_1+x_2)} \leq \frac{4}{3}$	I_2 $(x_1+x_2,\ x_3)$
E_8	$\frac{1}{2}$ $\frac{1}{2}$ $\frac{1}{2}$ 1 1 1	$\frac{\frac{1}{2}[2-(x_1+x_2+x_3)]+(x_1+x_2+x_3-1)}{\frac{1}{2}(x_1+x_2+x_3-x_1x_2)} = \frac{\frac{1}{2}(x_1+x_2+x_3)}{\frac{1}{2}(x_1+x_2+x_3-x_1x_2)} \leq \frac{4}{3}$	$\frac{\frac{1}{2}[2-(x_1+x_2+x_3)]+(x_1+x_2+x_3-1)}{\frac{1}{2}(x_1+x_2+x_3-x_1x_2)} = \frac{\frac{1}{2}(x_1+x_2+x_3)}{\frac{1}{2}(x_1+x_2+x_3-x_1x_2)} \leq \frac{4}{3}$	$\frac{\frac{1}{2}[2-(x_1+x_2+x_3)]+(x_1+x_2+x_3-1)}{\frac{1}{2}(x_1+x_2+x_3-x_1x_2)} = \frac{\frac{1}{2}(x_1+x_2+x_3)}{\frac{1}{2}(x_1+x_2+x_3-x_1x_2)} \leq \frac{4}{3}$	I_1 (x_1,x_2)
	Feasibility	$f \in \mathcal{F}_3^1$	$f \in \mathcal{F}_3^2$	$f \in \mathcal{F}_3^3$	

Table 1: Proof of 4/3 bound for region R_2

$\frac{\overline{f}^{+}(\boldsymbol{x})}{\underline{f}^{++}(\boldsymbol{x})}$	Extreme point	D_2 $\left[\overline{f}^{+}(\boldsymbol{x}) = f(\{1\})x_1 + f(\{2\})(1-x_1-x_3) + f(\{3\})(1-x_1-x_2) + f(\{2,3\})(x_1+x_2+x_3-1)\right]$	D_6 $\left[\overline{f}^{+}(\boldsymbol{x}) = f(\{2\})(1-x_3) + f(\{3\})(1-x_1-x_2) + f(\{1,3\})x_1 + f(\{2,3\})(x_2+x_3-1)\right]$	D_8 $\left[\overline{f}^{+}(\boldsymbol{x}) = f(\{2\})(1-x_1-x_3) + f(\{3\})(1-x_2) + f(\{1,2\})x_1 + f(\{2,3\})(x_2+x_3-1)\right]$	Inequality used (α, β in brackets)
E_1	1 0 0 1 1 0	$\frac{x_1}{x_1} \le \frac{4}{3}$	$\frac{x_1}{x_1} \le \frac{4}{3}$	$\frac{x_1}{x_1} \le \frac{4}{3}$	
E_2	0 1 0 1 0 1	$\frac{x_2}{x_2} \le \frac{4}{3}$	$\frac{x_2}{x_2} \le \frac{4}{3}$	$\frac{(1-x_1-x_3)+x_1+(x_2+x_3-1)}{x_2} = \frac{x_2}{x_2} \le \frac{4}{3}$	
E_3	0 0 1 0 1 1	$\frac{x_3}{x_3} \le \frac{4}{3}$	$\frac{x_3}{x_3} \le \frac{4}{3}$	$\frac{(1-x_2)+(x_2+x_3-1)}{x_3} = \frac{x_3}{x_3} \le \frac{4}{3}$	
E_4	1 1 0 1 1 1	$\frac{1-(1-x_1-x_2)}{x_1+x_2-x_1x_2} = \frac{x_1+x_2}{x_1+x_2-x_1x_2} \le \frac{4}{3}$	$\frac{1-(1-x_1-x_2)}{x_1+x_2-x_1x_2} = \frac{x_1+x_2}{x_1+x_2-x_1x_2} \le \frac{4}{3}$	*N.A.*	I_1 (x_1, x_2)
E_5	1 0 1 1 1 1	$\frac{1-(1-x_1-x_3)}{x_1+x_3-x_1x_3} = \frac{x_1+x_3}{x_1+x_3-x_1x_3} \le \frac{4}{3}$	*N.A.*	$\frac{1-(1-x_1-x_3)}{x_1+x_3-x_1x_3} = \frac{x_1+x_3}{x_1+x_3-x_1x_3} \le \frac{4}{3}$	I_1 (x_1, x_3)
E_6	0 1 1 1 1 1	*N.A.*	$\frac{1}{x_2+x_3-x_2x_3} \le \frac{4}{3}$	$\frac{1}{x_2+x_3-x_2x_3} \le \frac{4}{3}$	I_2 (x_2, x_3)
E_7	1 1 1 1 1 1	$\frac{1}{(x_1+x_2)+x_3-x_3(x_1+x_2)} \le \frac{4}{3}$	$\frac{1}{(x_1+x_2)+x_3-x_3(x_1+x_2)} \le \frac{4}{3}$	$\frac{1}{(x_1+x_2)+x_3-x_3(x_1+x_2)} \le \frac{4}{3}$	I_2 $(x_1+x_2,\ x_3)$
E_8	$\frac{1}{2}$ $\frac{1}{2}$ $\frac{1}{2}$ 1 1 1	$\frac{\frac{1}{2}[2-(x_1+x_2+x_3)]+(x_1+x_2+x_3-1)}{\frac{1}{2}(x_1+x_2+x_3-x_1x_2)} = \frac{\frac{1}{2}(x_1+x_2+x_3)}{\frac{1}{2}(x_1+x_2+x_3-x_1x_2)} \le \frac{4}{3}$	$\frac{\frac{1}{2}[2-(x_1+x_2+x_3)]+(x_1+x_2+x_3-1)}{\frac{1}{2}(x_1+x_2+x_3-x_1x_2)} = \frac{\frac{1}{2}(x_1+x_2+x_3)}{\frac{1}{2}(x_1+x_2+x_3-x_1x_2)} \le \frac{4}{3}$	$\frac{\frac{1}{2}[2-(x_1+x_2+x_3)]+(x_1+x_2+x_3-1)}{\frac{1}{2}(x_1+x_2+x_3-x_1x_2)} = \frac{\frac{1}{2}(x_1+x_2+x_3)}{\frac{1}{2}(x_1+x_2+x_3-x_1x_2)} \le \frac{4}{3}$	I_1 (x_1, x_2)
	Feasibility	$f \in \mathcal{F}_3^1$	$f \in \mathcal{F}_3^2$	$f \in \mathcal{F}_3^3$	

Table 2: Proof of 4/3 bound for region R_3

$\frac{\overline{f}^{+}(\boldsymbol{x})}{\underline{f}^{++}(\boldsymbol{x})}$	Extreme point	D_5 $\left[\overline{f}^{+}(\boldsymbol{x}) = \begin{array}{c} f(\{1\})(1-x_3) + f(\{3\})(1-x_1-x_2) \\ +f(\{1,3\})(x_1+x_3-1) + f(\{2,3\})x_2 \end{array}\right]$	D_6 $\left[\overline{f}^{+}(\boldsymbol{x}) = \begin{array}{c} f(\{2\})(1-x_3) + f(\{3\})(1-x_1-x_2) \\ +f(\{1,3\})x_1 + f(\{2,3\})(x_2+x_3-1) \end{array}\right]$	D_7 $\left[\overline{f}^{+}(\boldsymbol{x}) = \begin{array}{c} f(\{3\})(2-x_1-x_2-x_3) + f(\{1,2\})(1-x_3) \\ +f(\{1,3\})(x_1+x_3-1) + f(\{2,3\})(x_2+x_3-1) \end{array}\right]$	Inequality used (α, β in brackets)
E_1	1 0 0 1 1 0	$\frac{(1-x_3)+(x_1+x_3-1)}{x_1} = \frac{x_1}{x_1} \leq \frac{4}{3}$	$\frac{x_1}{x_1} \leq \frac{4}{3}$	$\frac{(1-x_3)+(x_1+x_3-1)}{x_1} = \frac{x_1}{x_1} \leq \frac{4}{3}$	
E_2	0 1 0 1 0 1	$\frac{x_2}{x_2} \leq \frac{4}{3}$	$\frac{(1-x_3)+(x_2+x_3-1)}{x_2} = \frac{x_2}{x_2} \leq \frac{4}{3}$	$\frac{(1-x_3)+(x_2+x_3-1)}{x_2} = \frac{x_2}{x_2} \leq \frac{4}{3}$	
E_3	0 0 1 0 1 1	$\frac{(1-x_1-x_2)+(x_1+x_3-1)+x_2}{x_3} = \frac{x_3}{x_3} \leq \frac{4}{3}$	$\frac{(1-x_1-x_2)+x_1+(x_2+x_3-1)}{x_3} = \frac{x_3}{x_3} \leq \frac{4}{3}$	$\frac{(2-x_1-x_2-x_3)+(x_1+x_3-1)+(x_2+x_3-1)}{x_3} = \frac{x_3}{x_3} \leq \frac{4}{3}$	
E_4	1 1 0 1 1 1	$\frac{1-(1-x_1-x_2)}{x_1+x_2-x_1x_2} = \frac{x_1+x_2}{x_1+x_2-x_1x_2} \leq \frac{4}{3}$	$\frac{1-(1-x_1-x_2)}{x_1+x_2-x_1x_2} = \frac{x_1+x_2}{x_1+x_2-x_1x_2} \leq \frac{4}{3}$	$N.A.$	I_1 (x_1, x_2)
E_5	1 0 1 1 1 1	$\frac{1}{x_1+x_3-x_1x_3} \leq \frac{4}{3}$	$N.A.$	$\frac{1}{x_1+x_3-x_1x_3} \leq \frac{4}{3}$	I_2 (x_1, x_3)
E_6	0 1 1 1 1 1	$N.A.$	$\frac{1}{x_2+x_3-x_2x_3} \leq \frac{4}{3}$	$\frac{1}{x_2+x_3-x_2x_3} \leq \frac{4}{3}$	I_2 (x_2, x_3)
E_7	1 1 1 1 1 1	$\frac{1}{(x_1+x_2)+x_3-x_3(x_1+x_2)} \leq \frac{4}{3}$	$\frac{1}{(x_1+x_2)+x_3-x_3(x_1+x_2)} \leq \frac{4}{3}$	$\frac{1}{(x_1+x_2)+x_3-x_3(x_1+x_2)} \leq \frac{4}{3}$	I_2 $(x_1+x_2,\ x_3)$
E_8	$\frac{1}{2}$ $\frac{1}{2}$ $\frac{1}{2}$ 1 1 1	$\frac{\frac{1}{2}[2-(x_1+x_2+x_3)]+(x_1+x_2+x_3-1)}{\frac{1}{2}(x_1+x_2+x_3-x_1x_2)} = \frac{\frac{1}{2}(x_1+x_2+x_3)}{\frac{1}{2}(x_1+x_2+x_3-x_1x_2)} \leq \frac{4}{3}$	$\frac{\frac{1}{2}[2-(x_1+x_2+x_3)]+(x_1+x_2+x_3-1)}{\frac{1}{2}(x_1+x_2+x_3-x_1x_2)} = \frac{\frac{1}{2}(x_1+x_2+x_3)}{\frac{1}{2}(x_1+x_2+x_3-x_1x_2)} \leq \frac{4}{3}$	$\frac{\frac{1}{2}[2-(x_1+x_2+x_3)]+(x_1+x_2+x_3-1)}{\frac{1}{2}(x_1+x_2+x_3-x_1x_2)} = \frac{\frac{1}{2}(x_1+x_2+x_3)}{\frac{1}{2}(x_1+x_2+x_3-x_1x_2)} \leq \frac{4}{3}$	I_1 (x_1, x_2)
	Feasibility	$f \in \mathcal{F}_3^1$	$f \in \mathcal{F}_3^2$	$f \in \mathcal{F}_3^3$	

Table 3: Proof of 4/3 bound for region R_4

$\dfrac{\overline{f}^{+}(\boldsymbol{x})}{\underline{f}^{++}(\boldsymbol{x})}$	Extreme point	D_9 $\left[\overline{f}^{+}(\boldsymbol{x}) = \begin{array}{r} f(\{1,2\})(1-x_3)+f(\{1,3\})(1-x_2)+f(\{2,3\})(1-x_1) \\ +f(\{1,2,3\})(x_1+x_2+x_3-2) \end{array}\right]$	Inequality used (α, β in brackets)
E_1	1 0 0 1 1 0	$\dfrac{(1-x_2)+(1-x_3)+(x_1+x_2+x_3-2)}{x_1(1-x_2)+x_1(1-x_3)+x_1(x_2+x_3-1)} = \dfrac{x_1}{x_1} \leq \dfrac{4}{3}$	
E_2	0 1 0 1 0 1	$\dfrac{(1-x_1)+(1-x_3)+(x_1+x_2+x_3-2)}{\left[\begin{array}{c} (1-x_3)(x_2-x_1)+(x_2x_3-x_1(x_2+x_3-1)) \\ +x_1(1-x_3)+x_1(x_2+x_3-1) \end{array}\right]} = \dfrac{x_2}{x_2} \leq \dfrac{4}{3}$	
E_3	0 0 1 0 1 1	$\dfrac{(1-x_1)+(1-x_2)+(x_1+x_2+x_3-2)}{\left[\begin{array}{c} (1-x_2)(x_3-x_1)+(x_2x_3-x_1(x_2+x_3-1)) \\ +x_1(1-x_2)+x_1(x_2+x_3-1) \end{array}\right]} = \dfrac{x_3}{x_3} \leq \dfrac{4}{3}$	
E_4	1 1 0 1 1 1	$\dfrac{1}{1-[(1-x_3)(1-x_2)+(1-x_2)(x_3-x_1)]} = \dfrac{1}{x_1+x_2-x_1x_2} \leq \dfrac{4}{3}$	I_2 $(x_1,\ x_2)$
E_5	1 0 1 1 1 1	$\dfrac{1}{1-[(1-x_3)(x_2-x_1)+(1-x_2)(1-x_3)]} = \dfrac{1}{x_1+x_3-x_1x_3} \leq \dfrac{4}{3}$	I_2 $(x_1,\ x_3)$
E_6	0 1 1 1 1 1	$\dfrac{1}{1-(1-x_2)(1-x_3)} = \dfrac{1}{x_2+x_3-x_2x_3} \leq \dfrac{4}{3}$	I_2 $(x_2,\ x_3)$
E_7	1 1 1 1 1 1	$\dfrac{1}{1-(1-x_2)(1-x_3)} = \dfrac{1}{x_2+x_3-x_2x_3} \leq \dfrac{4}{3}$	I_2 $(x_2,\ x_3)$
E_8	$\frac{1}{2}$ $\frac{1}{2}$ $\frac{1}{2}$ 1 1 1	$\dfrac{1}{1-\frac{1}{2}\left[\begin{array}{c} (1-x_2)(x_3-x_1) \\ +(1-x_3)(x_2-x_1)-(1-x_2)(1-x_3) \end{array}\right]} = \dfrac{1}{\left[x_1+\left(\dfrac{x_2+x_3}{2}\right)-x_1\left(\dfrac{x_2+x_3}{2}\right)\right]} \leq \dfrac{4}{3}$	I_2 $\left(x_1,\ \dfrac{x_2+x_3}{2}\right)$
	Feasibility	$f \in \mathcal{F}_3$	

Table 4: Proof of 4/3 bound for region R_5