Distributionally Robust Optimization

Problem Set 1 (Due Date: 8 July 2019)

1. Consider the following two stage stochastic optimization formulation for scheduling the appointments of a set of n patients. Specifically, let x_i be the amount of service time that is scheduled for patient i for $i = 1, \ldots, n-1$. Note that the patient 1 is scheduled to arrive at time 0, patient 2 at time x_1 and so on for patient n who arrives at time $x_1 + x_2 + \ldots + x_{n-1}$. The stochastic program is:

$$\begin{aligned} \min_{\boldsymbol{x}} \quad & E[Q(\boldsymbol{x}, \tilde{\boldsymbol{h}})] \\ \text{s.t.} \quad & x_i \geq 0, \qquad \forall i = 1, \ldots, n-1. \end{aligned}$$

where:

$$\begin{aligned} Q(\boldsymbol{x}, \boldsymbol{h}) = \min_{\boldsymbol{w}, \boldsymbol{s}} \quad & \sum_{i=1}^{n} c_w w_i + \sum_{i=1}^{n} c_s s_i \\ \text{s.t.} \quad & w_i \geq w_{i-1} + h_{i-1} - x_{i-1}, \quad \forall i = 2, \ldots, n \\ & s_i \geq -w_{i-1} - h_{i-1} + x_{i-1}, \quad \forall i = 2, \ldots, n \\ & w_1 = 0, s_1 = 0, \\ & w_i \geq 0, s_i \geq 0, \qquad \forall i = 2, \ldots, n, \end{aligned}$$

and $\tilde{h}_i$ is the random service time for patient i, w_i is the waiting time for patient i, s_i is the idle time of the server prior to serving patient i and c_w and c_s are the unit costs of waiting and idle times.

(a) Show that this stochastic optimization problem has complete recourse.

(b) You need to solve the following instances of the appointment scheduling problem using an optimization software. You can use any solver of your choice to do so (one example is JuMP in Julia which provides the modeling language which can be integrated with solvers such as Gurobi to solve linear and integer programs). Set $n = 10$ to be the number of patients and assume that the service times are random and independently distributed in the interval $[1, 3]$. Use a set of 5000 samples to solve the appointment scheduling problem for each of the following costs:

   (a) $c_s = 5$, $c_w = 5$

   (b) $c_s = 2$, $c_w = 8$

   (c) $c_s = 8$, $c_w = 2$

   Provide the optimal schedules and optimal expected costs you obtain from solving the problem. Provide the printout of the code that you used to solve the problem.

(c) Look at the structure of the optimal schedule. Is there any pattern you see in the optimal schedules?

2. Consider the decision version of the minimum sum of squares problem. This problem is given as follows:

   Input: A finite set $N = \{1, 2, \ldots, n\}$ with a nonnegative integer value c_i for each $i \in N$ and two positive integers k and m where $k \leq n$.

   Can the set N be partitioned into k disjoint sets $N_1, N_2, \ldots, N_k$ such that:

$$\sum_{j=1}^{k} \left(\sum_{i \in N_j} c_i \right)^2 \leq m?$$

   Prove that this problem is NP-complete (Hint: You can try a reduction from the PARTITION problem).

3. In the ellipsoid method, a key step is to find a rotation matrix R of size $n \times n$ that satisfies the following two properties:

   (a) $\boldsymbol{R}'\boldsymbol{R} = \boldsymbol{I}$

   (b) $\boldsymbol{R}\boldsymbol{u} = \|\boldsymbol{u}\|\boldsymbol{e}_1$

   where $\boldsymbol{I}$ is an identity matrix, $\boldsymbol{e}_1$ is the vector with 1 in the first entry and 0 otherwise and $\|\boldsymbol{u}\|$ is the 2-norm of the vector. Show that the matrix below satisfies the properties (a) and (b):

$$\boldsymbol{R} = 2\frac{(\boldsymbol{u} + \|\boldsymbol{u}\|\boldsymbol{e}_1)(\boldsymbol{u} + \|\boldsymbol{u}\|\boldsymbol{e}_1)'}{\|\boldsymbol{u} + \|\boldsymbol{u}\|\boldsymbol{e}_1\|^2} - \boldsymbol{I}$$

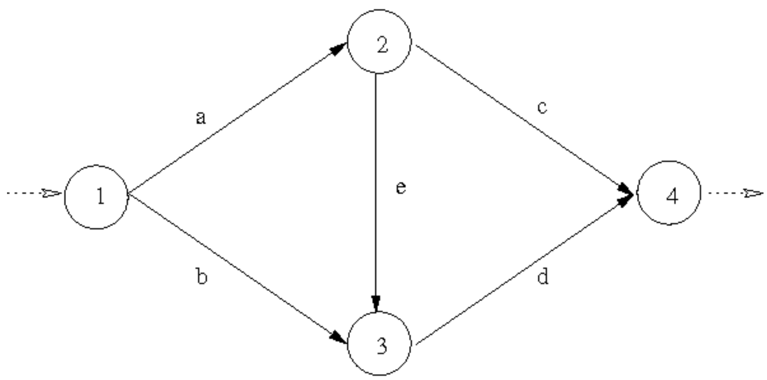
4. Consider a minimum directed spanning tree problem: Given a directed graph $G(V, E)$, a special vertex $r \in V$ and a positive integer cost c_{ij} for each edge $(i, j) \in E$, find a subgraph of minimum total cost that contains directed paths from r to all other vertices. The linear programming relaxation for the minimum directed spanning tree problem is given as:

$$\begin{aligned} \min \quad & \sum_{(i,j) \in E} c_{ij} x_{ij} & \\ \text{s.t.} \quad & \sum_{(i,j) \in E : i \in S, j \in V \setminus S} x_{ij} \geq 1, & \forall S \subset V, r \in S \\ & x_{ij} \geq 0, & \forall (i, j) \in E. \end{aligned}$$

   (a) Design a polynomial time separation oracle for this linear program.

   (b) Construct the dual of this linear program and provide an interpretation of strong duality for this problem.

5. Consider the following PERT network where the project makespan is given by the longest path from node 1 to 4. We are interested in estimating the worst-case expected project makespan where the distributions of the individual activity durations are known but the joint distribution is unknown.



   (a) Assume that the marginal distributions of the activity duration $\tilde{t}_a$, $\tilde{t}_b$, $\tilde{t}_c$, $\tilde{t}_d$ and $\tilde{t}_e$ are given by the set of values $\{1, 2, \ldots, 10\}$, such that it takes these values with uniform probability. Compute the worst-case expected project makespan. You will need to use an optimization solver for this problem.

   (b) Provide an estimate of the probability that each of the paths are critical (namely on the longest path) and the probability that each of the activities are critical from the worst-case distribution.

   (c) Assume that the activity durations are perfectly positively dependent. What is the expected project makespan in this case? Is this the worst-case distribution?

6. For a random vector $\tilde{\boldsymbol{c}} = (\tilde{c}_, \ldots, \tilde{c}_n)$ which takes values in the set $\mathcal{C}$, the entropy is defined as:

$$\text{Entropy} = -\sum_{\boldsymbol{c} \in \mathcal{C}} p(\boldsymbol{c}) \log p(\boldsymbol{c}),$$

where $p(\boldsymbol{c}) = P(\tilde{\boldsymbol{c}} = \boldsymbol{c})$. Show that the probability distribution of the random vector which maximizes the entropy, given fixed marginal distributions is given by the independent distribution. Hint: Use the Karush-Kuhn-Tucker conditions.

7. Let $c_i, i = 1, \ldots, n$ be nonnegative numbers. Show that the set function defined below for any $S \subseteq \{1, \ldots, n\}$:

$$f(S) = \frac{1}{2} \sum_{i \in \mathcal{S}} c_i^2 + \frac{1}{2} \left(\sum_{i \in \mathcal{S}} c_i \right)^2,$$

is supermodular.