

# Beyond Normality: A Cross Moment-Stochastic User Equilibrium Model\*

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## Abstract

The Stochastic User Equilibrium (SUE) model predicts traffic equilibrium flow assuming that users choose their perceived maximum utility paths (or perceived shortest paths) while accounting for the effects of congestion that arise due to users sharing links. Inspired by recent work on distributionally robust optimization, specifically a Cross Moment (CMM) choice model, we develop a new SUE model that uses the mean and covariance information on path utilities but does not assume the particular form of the distribution. Robustness to distributional assumptions is obtained in this model by minimizing the worst-case expected cost over all distributions with fixed two moments. We show that under mild conditions, the CMM-SUE (Cross Moment-Stochastic User Equilibrium) exists and is unique. By combining a simple projected gradient ascent method to evaluate path choice probabilities with a gradient descent method to find flows, we show that the CMM-SUE is efficiently computable. CMM-SUE provides both modeling flexibility and computational advantages over approaches such as the well-known MNP-SUE (Multinomial Probit-Stochastic User Equilibrium) model that require distributional (normality) assumptions to model correlation effects from overlapping paths. In particular, it avoids the use of simulation methods employed in computations for the distribution-based MNP-SUE model. Preliminary computational results indicate that CMM-SUE provides a practical distributionally robust alternative to MNP-SUE.

## 1 Introduction

Traffic assignment problems have intrigued system planners and researchers for several decades. Their key challenge is to develop models and methods that predict traffic flow by incorporating realistic phenomenon such as congestion effects and variations in driver route preferences among

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others. One of the popular traffic assignment concepts is the notion of a user equilibrium, which reflects the aggregate outcome of individual decisions made by drivers who choose routes from origin to destination nodes on the network. The simplest traffic equilibrium concept is the deterministic Wardropian User Equilibrium [see Wardrop [49]] which is based on the principle that:

*“At equilibrium, the travel costs on all routes that are actually used are equal to or less than those which would be experienced by a user on any unused route.”*

Users seek to minimize their own costs of transportation with the traffic flow that results from this principle leading to a User Equilibrium (commonly referred to as UE). At equilibrium, no user can reduce his/her own costs by unilaterally shifting from one route to another. Properties such as the existence of the equilibrium, the uniqueness of the equilibrium, and the efficient computability of the equilibrium have been shown using techniques from convex optimization [see Beckmann, McGuire and Winston [7]], variational inequalities [see Smith [47] and Dafermos [21]], and nonlinear complementarity problems [see Aashtiani and Magnanti [1]].

The Stochastic User Equilibrium (SUE) model introduced by Daganzo and Sheffi [24] generalizes the deterministic UE model by allowing users to have perceptions of travel costs that are different from the actual realized travel costs. Users choose routes based on their perceived travel costs, thus splitting up the aggregate demand for an origin-destination pair among the various paths connecting them. From a system planner’s view, the perceived travel costs are not observable and hence modeled as random variables with route choice probabilities representing the average fraction of users who move from the origin to the destination along particular paths. The SUE is based on the principle that:

*“At equilibrium, no driver can improve on his/her perceived travel cost by unilaterally changing routes.”*

In this paper, we employ a convex optimization approach to develop a new SUE model. The main novelty of the model is that the probability distribution of the perceived travel costs is itself uncertain in contrast to traditional SUE models where the probability distribution of the perceived travel costs is fixed. The structure of the paper and the main contributions are as follows:

- (a) In Section 2, we formally introduce the SUE model and provide a literature review on discrete choice models that are commonly used in traffic assignment problems.
- (b) In Section 3, we review the Cross Moment (CMM) choice model that has been recently introduced by Mishra et al. [36] and Ahipasaoglu, Li, and Natarajan [3]. In this model, the mean and the covariance matrix of the path utilities are assumed to be known, but the distribution itself is unknown. By focusing on a distribution that maximizes the expected perceived utility, the choice probabilities in the CMM model is computed by solving a convex optimization problem. We provide a simple network with two links for which the choice probabilities are found in closed form for the CMM model and contrast it with the Multinomial Probit (MNP) model. We also discuss a projected gradient ascent algorithm to find the choice probabilities in the CMM model that has been developed by Ahipasaoglu, Li, and Natarajan [3].
- (c) In Section 4, we develop a new stochastic user equilibrium model, referred to as the CMM-SUE model. The CMM-SUE model determines the equilibrium flows as the solution to a distributionally robust optimization problem where the worst-case expected cost in the optimization formulation of Sheffi and Powell [45] and Daganzo [23] is minimized. By accounting for the worst-case distribution, the equilibrium is robust from the system planner’s view. Under mild conditions, we show that the equilibrium exists and is unique. In this section, we also provide a generalization of the logit-based entropy optimization formulation of Fisk [27] to the CMM-SUE model. As far as we know, no such generalization of Fisk’s optimization formulation is

known for the probit model.

- (d) In Section 5, we develop an algorithm to compute the CMM-SUE. The key novelty of the algorithm is in the use of a projected gradient ascent method to compute choice probabilities in contrast to the MNP-SUE model which uses simulation. One advantage of such an approach is that it avoids sampling errors that arise from simulation methods. Furthermore since our algorithm is completely optimization based, to enhance the efficiency in computing the equilibrium we can use the choice probabilities computed in a previous iteration to warm start the optimization problem in the next iteration.
- (e) In Section 6, we present numerical tests to compare the MNP-SUE and CMM-SUE model. We provide evidence ranging from small networks to large networks that indicate the CMM-SUE model effectively captures correlation information among paths while being efficiently computable in comparison to the simulation based MNP-SUE model. The results indicate that CMM-SUE model provides a practical alternative to the MNP-SUE model. Concluding remarks are provided in Section 7.

## 2 Stochastic User Equilibrium

### 2.1 Notation

Throughout this paper, we use standard letters such as  $x$  to denote scalars, bold letters such as  $\mathbf{x}$  to denote vectors, bold capital letters such as  $\mathbf{X}$  to denote matrices, the tilde notation such as  $\tilde{x}$  to denote random variables, and the italic notation such as  $\mathcal{X}$  to denote sets (and  $|\mathcal{X}|$  to denote the size of the set). The  $n$ -dimensional Euclidean space is denoted by  $\mathfrak{R}_n$  and the nonnegative orthant by  $\mathfrak{R}_n^+$ . The transpose of a column vector  $\mathbf{x} \in \mathfrak{R}_n$  is denoted by  $\mathbf{x}^T$  which is a row vector and  $\mathbf{e}$  is the vector of all ones. The trace of a square matrix  $\mathbf{X}$ , denoted by  $\text{trace}(\mathbf{X})$ , is the sum of the diagonal entries of the matrix. A diagonal matrix with the vector  $\mathbf{x}$  along the diagonal entries is denoted by  $\text{Diag}(\mathbf{x})$ , whereas  $\text{diag}(\mathbf{X})$  is the column vector formed by the diagonal elements of the matrix  $\mathbf{X}$ . For a symmetric matrix  $\mathbf{X}$ , we use  $\mathbf{X} \succ 0$  to denote that the matrix is positive definite and  $\mathbf{X} \succeq 0$  to denote that the matrix is positive semidefinite. For a matrix  $\mathbf{X} \succeq 0$ ,  $\mathbf{X}^{1/2}$  denotes the unique positive semidefinite square root of the matrix such that  $\mathbf{X}^{1/2}\mathbf{X}^{1/2} = \mathbf{X}$ . For a matrix  $\mathbf{X}$ ,  $\mathbf{X}^\dagger$  denotes the unique Moore-Penrose pseudoinverse of the matrix. A summary of the additional notation that will be introduced in the following sections is provided in Appendix A.

### 2.2 SUE Model

A traffic network is represented as a directed graph  $\mathcal{G} = (\mathcal{N}, \mathcal{A})$ , with a set  $\mathcal{N}$  of nodes (for example intersections) and a set  $\mathcal{A}$  of arcs (for example roads). Associated with the network is a set of fixed origin-destination pairs indexed by  $w \in \mathcal{W}$  where  $(r_w, s_w)$  denotes the  $w$ th origin-destination pair. Let  $\mathcal{K}_w$  denote a finite set of directed simple paths connecting the origin  $r_w$  with the destination  $s_w$ . The specification of the choice set  $\mathcal{K}_w$  is flexible in that it is possible to restrict attention to a subset of paths connecting  $r_w$  and  $s_w$  that is justified behaviorally. The reader is referred to Prato [43] for a review of choice set generation methods developed for traffic networks.  $\mathcal{K} = \bigcup_{w \in \mathcal{W}} \mathcal{K}_w$  denotes the set of all paths where  $|\mathcal{K}_w|$  denotes the number of paths connecting  $r_w$  with  $s_w$ , and  $|\mathcal{K}| = \sum_{w \in \mathcal{W}} |\mathcal{K}_w|$  denotes the total number of paths. Let  $d_w > 0$  be the demand associated with the  $w$ th origin-destination pair. Each path  $k \in \mathcal{K}_w$  is associated with a flow  $x_{kw}$  with the path flow vector denoted by  $\mathbf{x} = (x_{kw})_{k \in \mathcal{K}_w, w \in \mathcal{W}}$ . A feasible flow on the network is a path flow vector  $\mathbf{x}$

satisfying the constraints:

$$\begin{aligned} \sum_{k \in \mathcal{K}_w} x_{kw} &= d_w, \quad \forall w \in \mathcal{W} \\ x_{kw} &\geq 0, \quad \forall k \in \mathcal{K}_w, w \in \mathcal{W}. \end{aligned}$$

The flow along an arc  $a \in \mathcal{A}$  is given by the sum of flows on all the paths that contain arc  $a$ :

$$f_a = \sum_{w \in \mathcal{W}} \sum_{k \in \mathcal{K}_w: k \ni a} x_{kw},$$

where  $\mathbf{f} = (f_a)_{a \in \mathcal{A}}$  is the arc flow vector.

In the traffic network, for the  $w$ th origin-destination pair, users choose from the set of paths  $\mathcal{K}_w$ . The perceived utility along a path  $k \in \mathcal{K}_w$  is modeled as a random variable and given by:

$$\tilde{u}_{kw}(\mathbf{f}) = -c_{kw}(\mathbf{f}) + \tilde{\epsilon}_{kw}, \quad \forall k \in \mathcal{K}_w, w \in \mathcal{W},$$

where  $c_{kw}(\mathbf{f})$  is the deterministic cost associated with the path and  $\tilde{\epsilon}_{kw}$  is the random component associated with the path. The congestion effect in the traffic network is modeled by letting the deterministic cost of the path be flow dependent (particularly a non-decreasing function of the flow value). To model the cost, associated with any arc  $a \in \mathcal{A}$  is a deterministic link cost given by  $c_a(f_a)$  which is assumed to be a non-decreasing, continuous function of the flow value. For a given arc flow vector  $\mathbf{f}$ , the deterministic cost associated with the path is then given by:

$$c_{kw}(\mathbf{f}) = \sum_{a \in \mathcal{A}: k \ni a} c_a(f_a), \quad \forall k \in \mathcal{K}_w, w \in \mathcal{W}.$$

The total cost vector is denoted by  $\mathbf{c}(\mathbf{f}) = (c_{kw}(\mathbf{f}))_{k \in \mathcal{K}_w, w \in \mathcal{W}}$ . Given realizations of the perceived route utilities, users selects the routes with maximum utility. The probability that a route  $k \in \mathcal{K}_w$  is chosen as the most preferred one for the  $w$ th origin-destination pair is flow dependent and given by:

$$p_{kw}(\mathbf{f}) = \mathbb{P}_{\theta_w} \left( -c_{kw}(\mathbf{f}) + \tilde{\epsilon}_{kw} \geq -c_{lw}(\mathbf{f}) + \tilde{\epsilon}_{lw}, \quad \forall l \neq k, l \in \mathcal{K}_w \right), \quad (2.1)$$

where  $\theta_w$  is the probability distribution of the random vector  $\tilde{\epsilon}_w = (\tilde{\epsilon}_{kw})_{k \in \mathcal{K}_w}$ . Associated with the choice model is the expected perceived utility where the expectation is taken over the distribution of the random components and given by a multidimensional integral of the form:

$$\mathbb{E}_{\theta_w} \left( \max_{k \in \mathcal{K}_w} (-c_{kw}(\mathbf{f}) + \tilde{\epsilon}_{kw}) \right) = \int_{\epsilon_w} \max_{k \in \mathcal{K}_w} (-c_{kw}(\mathbf{f}) + \epsilon_{kw}) \theta_w(\epsilon_w) d\epsilon_w. \quad (2.2)$$

with the maximum term in the integral defined as:

$$\max_{k \in \mathcal{K}_w} (-c_{kw}(\mathbf{f}) + \epsilon_{kw}) = \begin{cases} -c_{1w}(\mathbf{f}) + \epsilon_{1w} & \text{if } 1 = \arg \max_{k \in \mathcal{K}_w} (-c_{kw}(\mathbf{f}) + \epsilon_{kw}), \\ \vdots & \\ -c_{|\mathcal{K}_w|w}(\mathbf{f}) + \epsilon_{|\mathcal{K}_w|w} & \text{if } |\mathcal{K}_w| = \arg \max_{k \in \mathcal{K}_w} (-c_{kw}(\mathbf{f}) + \epsilon_{kw}), \end{cases}$$

This brings us to the definition of a Stochastic User Equilibrium.

**Definition 2.1** *In a SUE model, the equilibrium arc flow vector  $\mathbf{f}$  is the solution to the fixed point equation:*

$$f_a = \sum_{w \in \mathcal{W}} d_w \sum_{k \in \mathcal{K}_w: k \ni a} p_{kw}(\mathbf{f}), \quad \forall a \in \mathcal{A}, \quad (2.3)$$

where for each  $w \in \mathcal{W}$ , the choice probability vector  $\mathbf{p}_w(\mathbf{f}) = (p_{kw}(\mathbf{f}))_{k \in \mathcal{K}_w}$  is defined in (2.1) and lies in a unit simplex:

$$\mathcal{S}_w = \left\{ \mathbf{p}_w \in \mathfrak{R}_{|\mathcal{K}_w|} \mid \mathbf{e}^T \mathbf{p}_w = 1, \mathbf{p}_w \geq \mathbf{0} \right\}. \quad (2.4)$$

The corresponding flow for a path is given by:

$$x_{kw} = d_w \mathbb{P}_{\theta_w} \left( -c_{kw}(\mathbf{f}) + \tilde{\epsilon}_{kw} \geq -c_{lw}(\mathbf{f}) + \tilde{\epsilon}_{lw}, \quad \forall l \neq k, l \in \mathcal{K}_w \right), \quad \forall k \in \mathcal{K}_w, w \in \mathcal{W}. \quad (2.5)$$

In the next section, we review discrete choice models that are commonly used in traffic equilibrium problems.

### 2.3 Discrete Choice Models in SUE

The earliest application of a discrete choice model in transportation was developed with the Multinomial Logit (MNL) model (see Dial [26], Fisk [27]). The logit choice probabilities are obtained by assuming that  $\tilde{\epsilon}_{kw}$  are independent and identically distributed Gumbel random variables with a parameter  $\beta \geq 0$ . The choice probabilities in the logit model is given by the expression:

$$p_{kw}^{\text{mnl}}(\mathbf{f}) = \frac{e^{-\beta c_{kw}(\mathbf{f})}}{\sum_{l \in \mathcal{K}_w} e^{-\beta c_{lw}(\mathbf{f})}}, \quad \forall k \in \mathcal{K}_w, w \in \mathcal{W}.$$

For the MNL-SUE model, Fisk [27] showed that the optimal solution to the following convex optimization formulation with an entropy type objective function provides the equilibrium flows:

$$\begin{aligned} \min_{\mathbf{x}, \mathbf{f}} \quad & \sum_{a \in \mathcal{A}} \int_0^{f_a} c_a(t) dt + \frac{1}{\beta} \sum_{w \in \mathcal{W}} \sum_{k \in \mathcal{K}_w} x_{kw} \log x_{kw} \\ \text{s.t.} \quad & \sum_{k \in \mathcal{K}_w} x_{kw} = d_w, & \forall w \in \mathcal{W}, \\ & x_{kw} \geq 0, & \forall k \in \mathcal{K}_w, w \in \mathcal{W}, \\ & f_a = \sum_{w \in \mathcal{W}} \sum_{k \in \mathcal{K}_w: k \ni a} x_{kw}, & \forall a \in \mathcal{A}. \end{aligned} \quad (2.6)$$

Note that since the objective function is strictly convex and the minimization is over a compact convex set, the optimal solution is unique. One of the advantages of using the logit model in traffic assignment is that it is possible to develop efficient link-based algorithms to compute the equilibrium flows [see Akamatsu [5]]. These methods typically assume that either (a) the choice set is restricted to a set of “efficient” paths, namely all arcs along the path leading the user closer to the destination and further away from the origin [see Dial [26]], (b) the choice set consists of all cyclic (possibly infinite) and acyclic paths [see Bell [10], Akamatsu [4]] or (c) the network is a directed acyclic graph [see Bing-Feng et al. [15]]. However assumptions (a) and (b) are not always justified behaviorally while assumption (c) is restrictive for traffic networks. More importantly in the MNL model the utilities of routes are uncorrelated which is difficult to justify with overlapping routes.

As an alternative, Daganzo and Sheffi [24] proposed the use of a Multinomial Probit (MNP) model in traffic assignment to capture correlation information. In the MNP model, the vector of error terms  $\tilde{\epsilon}_w$  is assumed to be a multivariate normal random vector with mean  $\mathbf{0}$  and covariance

matrix  $\Sigma_w \succ 0$ . Simulation is used to evaluate the choice probabilities in MNP. An optimization formulation to find the equilibrium flow values was developed in the works of Daganzo and Sheffi [24], Daganzo [23] and Sheffi and Powell [45]. In terms of the arc flow decision variables, the unconstrained optimization formulation developed in Sheffi and Powell [45] is given as:

$$\min_{\mathbf{f}} \sum_{w \in \mathcal{W}} d_w \mathbb{E}_{\theta_w} \left( \max_{k \in \mathcal{K}_w} (-c_{kw}(\mathbf{f}) + \tilde{\epsilon}_{kw}) \right) + \sum_{a \in \mathcal{A}} f_a c_a(f_a) - \sum_{a \in \mathcal{A}} \int_0^{f_a} c_a(t) dt. \quad (2.7)$$

For multivariate normal distributions  $\theta_w, w \in \mathcal{W}$ , the optimal solution to (2.7) provides the equilibrium arc flows in the MNP-SUE model. An alternative formulation for the problem in terms of the arc cost variables was developed in Daganzo [23] and Sheffi and Powell [45]. In terms of arc cost variables, the formulation is given by:

$$\min_{\mathbf{c}} \sum_{w \in \mathcal{W}} d_w \mathbb{E}_{\theta_w} \left( \max_{k \in \mathcal{K}_w} \left( - \sum_{a \in \mathcal{A}: k \ni a} c_a + \tilde{\epsilon}_{kw} \right) \right) + \sum_{a \in \mathcal{A}} \int_{c_a}^{c_a} f_a(c) dc, \quad (2.8)$$

where  $c_a = c_a(0)$  is the free flow travel cost. To solve the optimization formulation for MNP-SUE, the Method of Successive Averages (MSA) [see Powell and Sheffi [41]] in conjunction with Monte-Carlo based simulations was proposed by Sheffi and Powell [45]. In the MSA, step-sizes are chosen in a predetermined sequence to avoid the computational cost of finding optimal step-sizes at each iteration of the algorithm. Convergence of the method was proved in [41, 45]. Maher and Hughes [35] proposed the use of Clark's approximation for probit choice probabilities to solve the MNP-SUE model approximately. Liu and Meng [33] have recently proposed a distributed computing approach to solve large scale probit based user equilibrium problems. Computation of the MNP-SUE on realistic traffic networks remains a challenge primarily due to the requirement of implementing a large scale simulation based optimization approach. In this paper, we develop a new class of SUE models where the covariance matrix is specified but the assumption of normality is dropped.

Generalizations of logit type choice probability formulas such as the C-logit model [see Cascetta et al. [18]], Xu and Chen [50]] and the Path-Size logit model [see Ben-Akiva and Bierlaire [12]] have also been used in traffic equilibrium problems. Path-based algorithms for SUE models with both C-logit [see Zhong et al. [53], Xu et al. [51]] and Path-Size logit [see Bovy et al. [16]] have been developed based on a generalization of Fisk's formulation in (2.6). Other variants of logit-based choice models using Generalized Extreme Value (GEV) models have also been proposed in the literature. One such example is the Nested Logit model [see Ben-Akiva and Lerman [13]] and the corresponding SUE model [see Bekhor and Prashker [8]]. While these models maintain the simplicity of logit type choice probability formulas, none of these models capture general correlation structures as in the MNP model. Another choice model that has received some attention recently is the Multinomial Weibit (MNW) model introduced by Castillo et al. [19], which also provides closed-form formulae for the route choice probabilities. In this model,  $\tilde{\epsilon}_{kw}$  are independent but not necessarily identical Weibull random variables. This leads to a new equilibrium, referred to as the MNW-SUE, which can handle the route-specific perception variance and is therefore useful for heterogeneous travelers with different knowledge levels of network conditions and trip lengths. Interested readers are referred to Kitthamkesorn and Chen [28, 29] for recent developments of the MNW-SUE model and its extensions, and Yao and Chen [52] for a discussion on the comparison of MNL-SUE and MNW-SUE models under the Braess Paradox.

It is important to note that an alternate stream of research in traffic equilibrium deals with uncertain travel costs. In this line of work, the travel times are assumed to be random and users select their routes using a risk averse criterion. The equilibrium is generated by the aggregate outcome of decentralized risk averse decisions. Examples include the stochastic Wardrop equilibrium where

users select routes by minimizing the sum of the mean and variance of travel costs [see Uchida and Iida [48], Bell and Cassir [11], Nikolova and Stier-Moses [39]], the percentile equilibrium where users select routes by minimizing a percentile of travel cost [see Ordóñez and Stier-Moses [40], Nie [38]], and the robust Wardrop equilibrium where users select routes by solving a robust optimization problem that imposes a limit on the number of arcs that can deviate from the mean [see Bertsimas and Sim [14], Ordóñez and Stier-Moses [40]]. Note that in this class of models the travel costs are random while in SUE models the perceived travel costs are random. Our focus in this paper is on SUE models.

### 3 Cross Moment Choice Model

Mishra et al. [36] and Ahipasaoglu, Li, and Natarajan [3] have recently introduced a new class of discrete choice models referred to as the Cross Moment (CMM) choice model using the concept of distributionally robustness. In contrast to standard discrete choice models where the distribution  $\theta_w$  of the random vector  $\tilde{\epsilon}_w$  is completely specified, in distributionally robust models the exact distribution is unknown. Rather the distribution is only known to lie in a set  $\theta_w \in \Theta_w$ . In the CMM model, the set  $\Theta_w(\mathbf{0}, \Sigma_w)$  is defined as the set of all probability distributions of  $\tilde{\epsilon}_w$  with mean  $\mathbb{E}_{\theta_w}[\tilde{\epsilon}_w] = \mathbf{0}$  and covariance matrix  $\text{Cov}_{\theta_w}[\tilde{\epsilon}_w] = \Sigma_w$ . Unlike the probit model, the CMM model makes no assumption of normality. The system planner computes the choice probabilities for an extremal distribution from the set  $\Theta_w(\mathbf{0}, \Sigma_w)$  that maximizes the expected perceived utility. A formal description of the model is provided next. All the proofs for the results in this section can be found in Ahipasaoglu, Li, and Natarajan [3].

Consider the simplest traffic setting with costs that are flow independent, namely  $c_{kw}(\mathbf{f}) = c_{kw}$ . For the  $w$ th origin-destination pair, the system planner in the CMM model solves the following optimization problem:

$$Z_w^{\text{cmm}} = \max_{\theta_w \in \Theta_w(\mathbf{0}, \Sigma_w)} \mathbb{E}_{\theta_w} \left( \max_{k \in \mathcal{K}_w} (-c_{kw} + \tilde{\epsilon}_{kw}) \right). \quad (3.9)$$

Problem (3.9) is equivalent to finding a joint distribution for the random vector that maximizes the expected perceived utility subject to first two moment information. The outer maximization problem is over all joint distributions of the random vector that is consistent with the first two moment information while the inner maximization problem is the choice problem of an individual user. Clearly the normal distribution is a feasible distribution in (3.9). Mishra, Natarajan and Teo [36] have developed a convex semidefinite optimization formulation to find the maximal expected user utility  $Z_w^{\text{cmm}}$  and the corresponding choice probabilities for a distribution  $\theta_w^{\text{cmm}}$  that solves (3.9). Computationally solving large scale semidefinite programs is however a challenge. Ahipasaoglu, Li, and Natarajan [3] have developed an alternate but simpler representative agent formulation for the CMM model. We review this approach next as it forms the key step in our user equilibrium formulation.

In a representative agent model, the aggregate behavior of a set of users is described through the choices made by a single representative user who has a preference for diversity [see Anderson, Palma and Thisse [6]]. In the traffic setting, this corresponds to the representative agent selecting a mixed strategy on the set of routes where the probabilities can be interpreted as the average fraction of users that choose the individual routes (pure strategies). Ahipasaoglu, Li, and Natarajan [3] developed a representative agent version of the CMM model by reformulating the problem as a nonlinear concave maximization problem over the unit simplex. We provide the main result from [3] next.

**Proposition 3.1 (Ahipasaoglu, Li, and Natarajan [3])** Assume that  $\Sigma_w \succ 0$ .

- (i) The maximum expected perceived utility  $Z_w^{cmm}$  in the CMM model (3.9) is the optimal objective value of the problem:

$$(CMM) \quad Z_w^{cmm} = \max_{\mathbf{p}_w} -\mathbf{c}_w^T \mathbf{p}_w + \text{trace} \left( \left( \Sigma_w^{1/2} \mathbf{S}(\mathbf{p}_w) \Sigma_w^{1/2} \right)^{1/2} \right) \quad (3.10)$$

s.t.  $\mathbf{p}_w \in \mathcal{S}_w$ ,

where the representative agent chooses a probability vector  $\mathbf{p}_w = (p_{kw})_{k \in \mathcal{K}_w}$  from the feasible region defined by the unit simplex in (2.4). The matrix  $\mathbf{S}(\mathbf{p}_w)$  in the trace function in (3.10) is defined as  $\text{Diag}(\mathbf{p}_w) - \mathbf{p}_w \mathbf{p}_w^T$  with the entries of the matrix given by:

$$\mathbf{S}(\mathbf{p}_w) = \begin{pmatrix} p_{1w} - p_{1w}^2 & -p_{1w}p_{2w} & \cdots & -p_{1w}p_{|\mathcal{K}_w|w} \\ -p_{1w}p_{2w} & p_{2w} - p_{2w}^2 & \cdots & -p_{2w}p_{|\mathcal{K}_w|w} \\ \vdots & \vdots & \ddots & \vdots \\ -p_{1w}p_{|\mathcal{K}_w|w} & -p_{2w}p_{|\mathcal{K}_w|w} & \cdots & p_{|\mathcal{K}_w|w} - p_{|\mathcal{K}_w|w}^2 \end{pmatrix}.$$

- (ii) The optimal decision vector  $\mathbf{p}_w^{cmm}$  in (3.10) is unique and lies strictly in the interior of the simplex. Furthermore,  $p_{kw}^{cmm}$  is the probability that the route  $k \in \mathcal{K}_w$  is chosen as the most preferred one for a distribution  $\theta_w^{cmm}$  of the random vector  $\tilde{\epsilon}_w$  that maximizes the expected perceived utility in (3.9):

$$p_{kw}^{cmm} = \mathbb{P}_{\theta_w^{cmm}} \left( -c_{kw} + \tilde{\epsilon}_{kw} \geq -c_{lw} + \tilde{\epsilon}_{lw}, \quad \forall l \neq k, l \in \mathcal{K}_w \right). \quad (3.11)$$

Two implications of Proposition 3.1 are worth noting:

- (a) Since the multivariate normal distribution is a feasible distribution in formulation (3.9),  $Z_w^{cmm}$  provides an upper bound on the expected perceived utility in the MNP choice model. The choice probabilities and the expected perceived utilities are however computed in a completely different manner in the two models. In the MNP model, numerical integration techniques are used when the number of routes are small and simulation techniques are used when the number of routes are large. On the other hand, in the CMM model, an optimization problem is solved where the optimal decision variables are the choice probabilities and the optimal objective value is the expected perceived utility.
- (b) The objective function in (3.10) is strictly concave with respect to the decision variables. In conjunction with the convexity of the feasible region, this implies that the CMM model is solvable as a tractable convex optimization problem and furthermore that the choice probabilities are strictly unique.

We provide a simple two link example next to illustrate the choice probabilities in the CMM model and compare it with the MNP model.

### 3.1 A Simple Example with Two Links

Consider the traffic network in Figure 3.1 with a single origin destination pair with  $\mathcal{W} = \{1\}$  and two links associated with the origin-destination pair. In this case,  $\mathcal{K} = \mathcal{K}_1 = \{1, 2\}$ . The utility



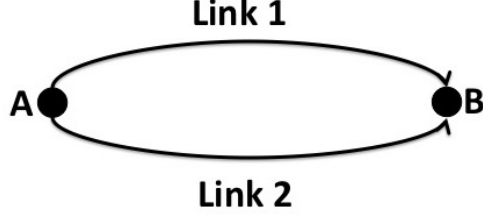


Figure 3.1: A two link network.

for link 1 is  $\tilde{u}_{11} = -c_1 + \tilde{\epsilon}_{11}$  and the utility for link 2 is  $\tilde{u}_{21} = -c_2 + \tilde{\epsilon}_{21}$ . The mean of the vector  $(\tilde{\epsilon}_{11}, \tilde{\epsilon}_{21})^T$  is zero and the covariance matrix is given as:

$$\Sigma_1 = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix},$$

where  $\sigma_1^2 = \text{Var}_{\theta_1}[\tilde{\epsilon}_{11}]$ ,  $\sigma_2^2 = \text{Var}_{\theta_1}[\tilde{\epsilon}_{21}]$  and  $\rho = \text{Corr}_{\theta_1}[\tilde{\epsilon}_{11}, \tilde{\epsilon}_{21}]$  with  $\rho \in (-1, 1)$ . The positive semidefinite square root of the two by two covariance matrix is given by the explicit formula:

$$\Sigma_1^{1/2} = \frac{1}{\sqrt{\sigma_1^2 + \sigma_2^2 + 2\sigma_1\sigma_2\sqrt{1-\rho^2}}} \begin{pmatrix} \sigma_1^2 + \sigma_1\sigma_2\sqrt{1-\rho^2} & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 + \sigma_1\sigma_2\sqrt{1-\rho^2} \end{pmatrix}.$$

The choice probability vector is  $\mathbf{p}_1 = (p_{11}, p_{21})^T$  with the matrix  $\mathbf{S}(\mathbf{p}_1)$  in (3.10) given by:

$$\begin{aligned} \mathbf{S}(\mathbf{p}_1) &= \begin{pmatrix} p_{11} - p_{11}^2 & -p_{11}p_{21} \\ -p_{11}p_{21} & p_{21} - p_{21}^2 \end{pmatrix}, \\ &= p_{11}(1 - p_{11}) \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \end{aligned}$$

where the second equality follows from  $p_{11} + p_{21} = 1$ . Substituting into formulation (3.10), we obtain an equivalent single variable optimization problem:

$$\begin{aligned} Z_1^{\text{cmm}} &= \max_{p_{11}} -c_1 p_{11} - c_2(1 - p_{11}) + \sqrt{p_{11}(1 - p_{11})} \sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2} \\ &\text{s.t. } 0 \leq p_{11} \leq 1. \end{aligned} \quad (3.12)$$

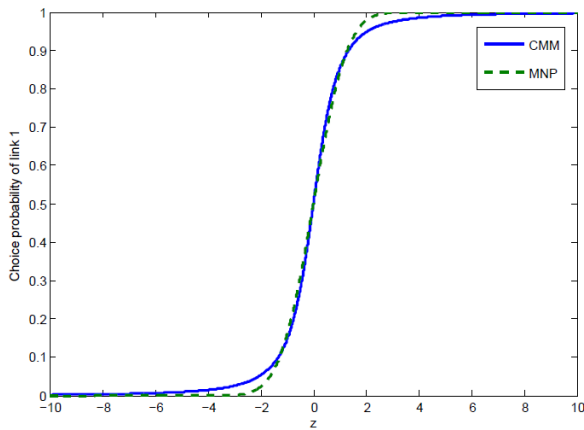
Solving for the first order optimality condition in (3.12), we find that:

$$p_{11}^{\text{cmm}} = \frac{1}{2} \left( 1 + \frac{-c_1 + c_2}{\sqrt{(-c_1 + c_2)^2 + \sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}} \right). \quad (3.13)$$

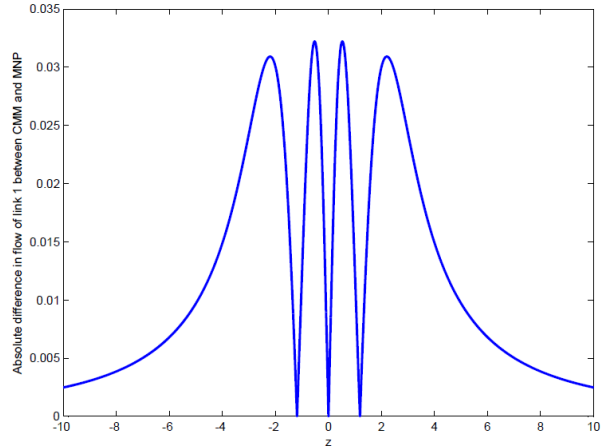
In comparison, the choice probability for link 1 in the MNP model is given as:

$$p_{11}^{\text{mnp}} = \Phi \left( \frac{-c_1 + c_2}{\sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}} \right), \quad (3.14)$$

where  $\Phi(\cdot)$  is the cumulative distribution function of the standard normal. The choice probability in the CMM model in (3.13) is an algebraic function of the parameters of the problem while the choice probability in the MNP model (3.14) is a transcendental function of the parameters of the problem illustrating a key difference in the nature of the formulas from the choice models<sup>1</sup>. In Figure 3.2 we compare the choice probabilities for the two link example. The choice probabilities



(a) Choice probability of link 1.



(b) Absolute difference in choice probability of link 1.

Figure 3.2: Choice probability in the two link network as a function of  $z = \frac{-c_1 + c_2}{\sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}}$ .

from the two models are fairly close across all values of means, variances, and correlation with the maximum absolute difference in the choice probability of a single link around 0.0321 (or 3.21%).

### 3.2 Projected Gradient Ascent Algorithm for CMM

While in the two link example it is possible to solve the CMM model in closed form, this is not the case for larger networks. To solve the optimization problem, Ahipasaoglu, Li, and Natarajan [3] proposed the use of a simple projected gradient method which we outline next. Denote the objective function in (3.10) as:

$$\mathbf{h}(\mathbf{p}_w) = -\mathbf{c}_w^T \mathbf{p}_w + \text{trace} \left( \left( \boldsymbol{\Sigma}_w^{1/2} \mathbf{S}(\mathbf{p}_w) \boldsymbol{\Sigma}_w^{1/2} \right)^{1/2} \right).$$

The directional derivative of the function  $\text{trace}((\boldsymbol{\Sigma}_w^{1/2} \mathbf{S}(\mathbf{p}_w) \boldsymbol{\Sigma}_w^{1/2})^{1/2})$  along the vector  $\mathbf{v}$  in the tangent space of the simplex is given as  $\mathbf{g}(\mathbf{p}_w)^T \mathbf{v}$  where:

$$\mathbf{g}(\mathbf{p}_w) = \frac{1}{2} \left( \text{diag} \left( \boldsymbol{\Sigma}_w^{1/2} \left( \mathbf{T}^{1/2}(\mathbf{p}_w) \right)^\dagger \boldsymbol{\Sigma}_w^{1/2} \right) - 2 \boldsymbol{\Sigma}_w^{1/2} \left( \mathbf{T}^{1/2}(\mathbf{p}_w) \right)^\dagger \boldsymbol{\Sigma}_w^{1/2} \mathbf{p}_w \right),$$

and  $\mathbf{T}(\mathbf{p}_w) = \boldsymbol{\Sigma}_w^{1/2} \mathbf{S}(\mathbf{p}_w) \boldsymbol{\Sigma}_w^{1/2}$ . The optimality condition for (3.10) is then given by the following two sets of conditions:

- (i)  $\bar{\nabla} \mathbf{h}(\mathbf{p}_w^{\text{cmm}}) = (-\mathbf{c}_w + \mathbf{g}(\mathbf{p}_w^{\text{cmm}})) - \frac{\mathbf{e}^T (-\mathbf{c}_w + \mathbf{g}(\mathbf{p}_w^{\text{cmm}})) \mathbf{e}}{|\mathcal{K}_w|} \mathbf{e} = \mathbf{0}$ ,
- (ii)  $\mathbf{p}_w^{\text{cmm}} \in \mathcal{S}_w$ ,

where in (i),  $\bar{\nabla} \mathbf{h}(\cdot)$  is the projected gradient of the objective function in the tangent space of the simplex which equals zero and in (ii), the probability vector lies in the unit simplex. The gradient

<sup>1</sup>An algebraic function is a function that is defined as the root of a polynomial equation in the variables. A function which is not algebraic is called a transcendental function. Examples of transcendental functions include exponential, logarithmic, and trigonometric functions.

method to solve (3.10) is provided in Algorithm 1.

**Input:** Parameters:  $\mathbf{c}_w, \Sigma_w$ , Starting point:  $\mathbf{p}_w^0$  in the interior of the simplex, Step length:  $\alpha$ , Tolerance:  $\epsilon$ , Constants:  $\tau \in (0, 1), \gamma \in (0, 1)$ .

**Output:** Solution  $\mathbf{p}_w$ .

$i \leftarrow 0$ ,

**while** the stopping criterion is violated, **do**

$\bar{\alpha} \leftarrow \alpha$   
 $\mathbf{p}_w^{(i+1)} \leftarrow \mathbf{p}_w^{(i)} + \bar{\alpha} \bar{\nabla} \mathbf{h}(\mathbf{p}_w^{(i)})$   
**while**  $\min_k p_{kw}^{(i+1)} \leq 0$  or  $\mathbf{h}(\mathbf{p}_w^{(i+1)}) - \mathbf{h}(\mathbf{p}_w^{(i)}) < \gamma \alpha \mathbf{g}(\mathbf{p}_w^{(i)})^T \bar{\nabla} \mathbf{h}(\mathbf{p}_w^{(i)})$  **do**  
   $\bar{\alpha} \leftarrow \tau \bar{\alpha}$   
   $\mathbf{p}_w^{(i+1)} \leftarrow \mathbf{p}_w^{(i)} + \bar{\alpha} \bar{\nabla} \mathbf{h}(\mathbf{p}_w^{(i)})$   
**end**  
 $i \leftarrow i + 1$   
**end**

**Algorithm 1:** Projected gradient ascent algorithm for  $\text{CMM}(\mathbf{c}_w, \Sigma_w)$ .

In Algorithm 1 we use the stopping criterion

$$\|\bar{\nabla} \mathbf{h}(\mathbf{p}_w^{(i)})\|_2 \leq \epsilon$$

to terminate the algorithm. Algorithm 1 uses a backtracking line search with a step size  $\alpha$ , and repeatedly shrinks it by a factor  $\tau$  until the choice probability vector is nonnegative and the Armijo rule is satisfied. The Armijo rule verifies if the move from the current feasible solution to the new solution achieves an adequate improvement in the objective based on the local gradient information. In the numerical implementation, the key difficulty in estimating the projected gradient  $\bar{\nabla} \mathbf{h}(\cdot)$  lies in the computation of the matrix square root and the Moore-Penrose pseudoinverse. To initialize the algorithm, a simple starting point is to use the equally weighted choice probability vector with  $\mathbf{p}_w^0 = \frac{1}{|\mathcal{K}_w|} \mathbf{e}$ .

## 4 A Cross Moment-Stochastic User Equilibrium Model

### 4.1 Distributionally Robust Optimization Formulation of CMM-SUE

In this section, we develop a new traffic equilibrium model under the assumption that the mean of the random vector  $\tilde{\mathbf{c}}_w$  is  $\mathbf{0}$  and the covariance matrix is  $\Sigma_w$  for  $w \in \mathcal{W}$  but the exact distribution is unknown. Our approach is based on the concept of distributionally robust optimization where the objective function is minimized with respect to the worst case distribution from a set of possible distributions. The system planner formulates the distributionally robust counterpart of (2.7) by minimizing the maximum objective function as follows:

$$\min_{\mathbf{f}} \sum_{w \in \mathcal{W}} d_w \max_{\theta_w \in \Theta_w(\mathbf{0}, \Sigma_w)} \mathbb{E}_{\theta_w} \left( \max_{k \in \mathcal{K}_w} (-c_{kw}(\mathbf{f}) + \tilde{c}_{kw}) \right) + \sum_{a \in \mathcal{A}} f_a c_a(f_a) - \sum_{a \in \mathcal{A}} \int_0^{f_a} c_a(t) dt, \quad (4.15)$$

where the minimization is with respect to the link flow variables and the maximization is with respect to probability distributions of the random vectors with fixed two moments. Equivalently the distributionally robust counterpart of (2.8) with arc cost variables is formulated as:

$$\min_{\mathbf{c}} \sum_{w \in \mathcal{W}} d_w \max_{\theta_w \in \Theta_w(\mathbf{0}, \Sigma_w)} \mathbb{E}_{\theta_w} \left( \max_{k \in \mathcal{K}_w} \left( - \sum_{a \in \mathcal{A}: k \ni a} c_a + \tilde{c}_{kw} \right) \right) + \sum_{a \in \mathcal{A}} \int_{c_a}^{c_a} f_a(c) dc. \quad (4.16)$$

We start by applying Proposition 3.1 to reformulate the inner maximization problem over probability distributions to a maximization problem over choice probabilities in the unit simplex. An equivalent formulation for (4.16) is hence given as:

$$\min_{\mathbf{c}} \sum_{w \in \mathcal{W}} d_w \max_{\mathbf{p}_w \in \mathcal{S}_w} \left( - \sum_{k \in \mathcal{K}_w} \left( \sum_{a \in \mathcal{A}: k \ni a} c_a \right) p_{kw} + \text{trace} \left( \left( \boldsymbol{\Sigma}_w^{1/2} \mathbf{S}(\mathbf{p}_w) \boldsymbol{\Sigma}_w^{1/2} \right)^{1/2} \right) \right) + \sum_{a \in \mathcal{A}} \int_{\underline{c}_a}^{c_a} f_a(c) dc.$$

Define the function:

$$\Phi(\mathbf{c}, \mathbf{p}) = \sum_{w \in \mathcal{W}} d_w \left( - \sum_{k \in \mathcal{K}_w} \left( \sum_{a \in \mathcal{A}: k \ni a} c_a \right) p_{kw} + \text{trace} \left( \left( \boldsymbol{\Sigma}_w^{1/2} \mathbf{S}(\mathbf{p}_w) \boldsymbol{\Sigma}_w^{1/2} \right)^{1/2} \right) \right) + \sum_{a \in \mathcal{A}} \int_{\underline{c}_a}^{c_a} f_a(c) dc.$$

We can rewrite the distributionally robust optimization problem in (4.16) as:

$$\min_{\mathbf{c}} \max_{\mathbf{p} \in \mathcal{S}} \Phi(\mathbf{c}, \mathbf{p}), \quad (4.17)$$

where the minimization is over the cost variables  $\mathbf{c}$  and the maximization is over the path choice probability vector  $\mathbf{p}$  that lies in the product of simplices  $\mathcal{S} = \mathcal{S}_1 \times \dots \times \mathcal{S}_{|\mathcal{W}|}$ . Clearly  $\Phi(\mathbf{c}, \mathbf{p})$  is convex with respect to the link cost variables  $\mathbf{c}$  and concave with respect to the path choice probabilities  $\mathbf{p}$ . Since the feasible region for both  $\mathbf{c}$  and  $\mathbf{p}$  is convex and compact for  $\mathbf{p}$  (product of the unit simplices) and the objective function is continuous, minmax equality holds (see Rockafellar [44], Sion [46]). Define a saddle point  $(\mathbf{c}^*, \mathbf{p}^*)$  of a function  $\Phi$  as:

$$\Phi(\mathbf{c}^*, \mathbf{p}) \leq \Phi(\mathbf{c}^*, \mathbf{p}^*) \leq \Phi(\mathbf{c}, \mathbf{p}^*) \quad \forall \mathbf{c}, \forall \mathbf{p} \in \mathcal{S}.$$

Then  $(\mathbf{c}^*, \mathbf{p}^*)$  is a saddle point that satisfies the minmax equality:

$$\min_{\mathbf{c}} \max_{\mathbf{p} \in \mathcal{S}} \Phi(\mathbf{c}, \mathbf{p}) = \Phi(\mathbf{c}^*, \mathbf{p}^*) = \max_{\mathbf{p} \in \mathcal{S}} \min_{\mathbf{c}} \Phi(\mathbf{c}, \mathbf{p}).$$

The optimality conditions for the problem (4.17) is given as:

$$\begin{aligned} - \sum_{w \in \mathcal{W}} d_w \sum_{k \in \mathcal{K}_w: k \ni a} p_{kw}^* + f_a(c_a^*) &= 0, \quad \forall a \in \mathcal{A}, \\ (-\mathbf{c}_w + \mathbf{g}(\mathbf{p}_w^*)) - \frac{\mathbf{e}^T (-\mathbf{c}_w + \mathbf{g}(\mathbf{p}_w^*)) \mathbf{e}}{|\mathcal{K}_w|} &= 0, \quad \forall w \in \mathcal{W}, \\ \mathbf{p}_w^* &\in \mathcal{S}_w, \quad \forall w \in \mathcal{W}. \end{aligned} \quad (4.18)$$

We now provide the formal definition of the CMM-SUE, where the equilibrium arc flow vector is characterized as the solution that satisfies the optimality conditions given in (4.18).

**Definition 4.1 (CMM-SUE)** *In the CMM-SUE model, the equilibrium arc flow vector  $\mathbf{f}^{cmm}$  is the solution to the fixed point equation:*

$$f_a^{cmm} = \sum_{w \in \mathcal{W}} d_w \sum_{k \in \mathcal{K}_w: k \ni a} p_{kw}^{cmm}(\mathbf{f}), \quad \forall a \in \mathcal{A}, \quad (4.19)$$

where for each  $w \in \mathcal{W}$ , the choice probability vector is computed for a distribution of  $\tilde{\boldsymbol{\epsilon}}_w$  with mean  $\mathbf{0}$  and covariance matrix  $\boldsymbol{\Sigma}_w \succ \mathbf{0}$  that maximizes the expected perceived utility as follows:

$$\mathbf{p}_w^{cmm}(\mathbf{f}) = \arg \max_{\mathbf{p}_w \in \mathcal{S}_w} -\mathbf{c}_w(\mathbf{f})^T \mathbf{p}_w + \text{trace} \left( \left( \boldsymbol{\Sigma}_w^{1/2} \mathbf{S}(\mathbf{p}_w) \boldsymbol{\Sigma}_w^{1/2} \right)^{1/2} \right). \quad (4.20)$$

The corresponding equilibrium flow for a path is given by:

$$x_{kw}^{cmm} = d_w p_{kw}^{cmm}(\mathbf{f}), \quad \forall k \in \mathcal{K}_w, w \in \mathcal{W}. \quad (4.21)$$

Under the assumption that the covariance matrix  $\Sigma_w \succ 0$  for all  $w \in \mathcal{W}$  and the link cost functions  $c_a(f_a)$  are non-decreasing and continuous, the optimal solution to the convex optimization problem in (4.20) is unique and continuous with respect to the cost function. Applying Brouwer's theorem, since a continuous function maps a compact convex set into itself, there exists a fixed point for the equilibrium. Furthermore, the choice probability for a path is strictly decreasing in the cost function of the path. Uniqueness of the user equilibrium follows (see Cantarella [17]). Note that in the optimization formulation, uniqueness follows from the observation that the objective function in (4.17) is strictly convex with respect to the  $\mathbf{c}$  variables and strictly concave with respect to the  $\mathbf{p}$  variables (see Proposition 3.1).

## 4.2 Extending Fisk's Formulation to CMM-SUE

In the previous section, we developed the CMM-SUE model as the solution to a minimax optimization problem. In this section, we develop an alternative optimization formulation for the CMM-SUE model that is inspired from the SUE formulation of Fisk [27] for the MNL choice model. Our main result is that the equilibrium link and path flows in (4.19)-(4.21) is the optimal solution to the formulation:

$$\begin{aligned}
\min_{\mathbf{x}, \mathbf{f}} \quad & \sum_{a \in \mathcal{A}} \int_0^{f_a} c_a(t) dt - \sum_{w \in \mathcal{W}} \text{trace} \left( \left( \Sigma_w^{1/2} (d_w \text{Diag}(\mathbf{x}_w) - \mathbf{x}_w \mathbf{x}_w^T) \Sigma_w^{1/2} \right)^{1/2} \right) \\
\text{s.t.} \quad & \sum_{k \in \mathcal{K}_w} x_{kw} = d_w, & \forall w \in \mathcal{W}, \\
& x_{kw} \geq 0, & \forall k \in \mathcal{K}_w, w \in \mathcal{W}, \\
& f_a = \sum_{w \in \mathcal{W}} \sum_{k \in \mathcal{K}_w: k \ni a} x_{kw}, & \forall a \in \mathcal{A}.
\end{aligned} \tag{4.22}$$

To show this, define a new set of variables  $\mathbf{p} = (p_{kw})_{k \in \mathcal{K}_w, w \in \mathcal{W}}$  as follows:

$$p_{kw} = x_{kw}/d_w.$$

Then, (4.22) is reformulated as:

$$\begin{aligned}
\min_{\mathbf{p}, \mathbf{f}} \quad & \sum_{a \in \mathcal{A}} \int_0^{f_a} c_a(t) dt - \sum_{w \in \mathcal{W}} d_w \text{trace} \left( \left( \Sigma_w^{1/2} (\text{Diag}(\mathbf{p}_w) - \mathbf{p}_w \mathbf{p}_w^T) \Sigma_w^{1/2} \right)^{1/2} \right) \\
\text{s.t.} \quad & \sum_{k \in \mathcal{K}_w} p_{kw} = 1, & \forall w \in \mathcal{W}, \\
& p_{kw} \geq 0, & \forall k \in \mathcal{K}_w, w \in \mathcal{W}, \\
& f_a = \sum_{w \in \mathcal{W}} d_w \sum_{k \in \mathcal{K}_w: k \ni a} p_{kw}, & \forall a \in \mathcal{A}.
\end{aligned}$$

The Karush-Kuhn-Tucker optimality conditions for this problem is given as:

$$\begin{aligned}
c_a(f_a) + \varphi_a &= 0, & \forall a \in \mathcal{A}, \\
-d_w \mathbf{g}(\mathbf{p}_w)_k + \lambda_w - d_w \sum_{a \in k} \varphi_a &= 0, & \forall k \in \mathcal{K}_w, w \in \mathcal{W}, \\
\mathbf{e}^T \mathbf{p}_w &= 1, & \forall w \in \mathcal{W}, \\
\mathbf{p}_w &\geq \mathbf{0}, & \forall w \in \mathcal{W}, \\
f_a - \sum_{w \in \mathcal{W}} d_w \sum_{k \in \mathcal{K}_w: k \ni a} p_{kw} &= 0, & \forall a \in \mathcal{A},
\end{aligned} \tag{4.23}$$

where  $\mathbf{g}(\mathbf{p}_w)_k$  is the  $k$ th coordinate of the gradient vector  $\mathbf{g}(\mathbf{p}_w)$ . Solving for the Lagrange multipliers, we get:

$$\varphi_a = -c_a(f_a) \quad \text{and} \quad \lambda_w = d_w \frac{\mathbf{e}^T (-\mathbf{c}_w + \mathbf{g}(\mathbf{p}_w))}{|\mathcal{K}_w|}.$$

If we substitute in (4.23), the optimality conditions reduce to the set of conditions in (4.18). This implies that the equilibrium arc flow vector in the CMM-SUE model is the solution to the optimization formulation in (4.22). Unlike the minimax formulation in the previous section, Formulation (4.22) is a straightforward minimization problem. Unlike the MNP model for which we are not aware of any straightforward extension of Fisk’s formulation, Formulation (4.22) provides such an extension for the CMM choice model.

## 5 Solution Algorithm for CMM-SUE

One of the well-known algorithms to find the arc flows of the MNP-SUE formulation is the Method of Successive Averages (MSA) developed by Sheffi and Powell [45]. MSA is a gradient-based method where in each step the current iterate is updated by adding a multiple of a descent direction. Finding a descent direction is not straightforward because it involves calculating path probabilities. Instead of calculating exact descent directions in each iteration, Sheffi and Powell [45] proposed to use directions obtained from a Monte-Carlo simulation. This provides unbiased estimates of the descent directions. The algorithm converges to the optimal solution almost surely under mild conditions. Maher [34] developed link-based algorithms for the logit SUE problem, using Sheffi and Powell’s formulation. Maher proposed an algorithm that uses the same search direction as the MSA algorithm, but calculates an approximately optimal step size in this direction, thus improving overall convergence. Two adaptations of the Davidon-Fletcher-Powell (DFP) method were also considered, but were found inferior to the above method. Another example is the entropy-based algorithm developed by Dial, which is specific for the logit route choice model. This algorithm exploits the fact that for the logit function, it is possible to map path flows from link flows and vice-versa. Dial’s and Maher’s algorithms exploit mathematical properties of the logit function to develop efficient link-based algorithms. Recent research on path-based algorithms has demonstrated and established that it is a viable approach for deterministic traffic assignment problems with reasonably large network size (see, for example [20]). Most of the work has been on two algorithms: the Disaggregated Simplicial Decomposition (DSD) algorithm and the gradient projection (GP) algorithm. The adaptation of these algorithms to the logit SUE are discussed by Bekhor and Toledo [9, 25].

In this section, we propose a simple descent algorithm to compute the equilibrium arc flows for the CMM-SUE model. The main difference between the MSA algorithm used for calculating the MNP-SUE and our algorithm is that we use optimization to find the choice probabilities rather than using simulation as is the case for MNP choice models. Therefore, we are able to find descent directions at all iterations of the algorithm. This guarantees that the algorithm converges to the equilibrium and the computational efficiency in each iteration is significantly improved.

As discussed, the CMM-SUE arc flows can be obtained from the optimization problem (4.17), which can be rewritten as:

$$\min_{\mathbf{c}} \max_{\mathbf{p} \in \mathcal{S}} \Phi(\mathbf{c}, \mathbf{p}) = \min_{\mathbf{f}} \Phi_f(\mathbf{f}), \quad (5.24)$$

in terms of the arc flow variables, where  $\Phi_f(\mathbf{f}) := \max_{\mathbf{p} \in \mathcal{S}} \Phi(\mathbf{c}(\mathbf{f}), \mathbf{p})$ . For fixed  $\mathbf{f}$ ,  $\Phi_f(\mathbf{f})$  is separable in  $|\mathcal{W}|$  subproblems, each one of which is equivalent to finding the optimal choice probabilities

in a CMM model, i.e., each subproblem corresponds to an origin-destination pair  $w \in \mathcal{W}$  and can be solved efficiently by applying Algorithm 1 with  $\mathbf{c}_w$  and  $\Sigma_w$ . Let  $\mathbf{p}^*(\mathbf{f}) := \arg \max_{\mathbf{p} \in \mathcal{S}} \Phi(\mathbf{c}(\mathbf{f}), \mathbf{p})$ . The partial derivative of  $\Phi_f(\mathbf{f})$  with respect to flow variable  $f_a$  is as follows:

$$\frac{\partial \Phi_f}{\partial f_a}(\mathbf{f}) = -\frac{\partial c_a(f_a)}{\partial f_a} \left( \sum_{w \in \mathcal{W}} \sum_{k \in \mathcal{K}_w: k \ni a} d_w(\mathbf{p}_w^*)_k - f_a \right).$$

Since  $c_a$  is an increasing function of flow  $f_a$ , the vector

$$\left( \sum_{w \in \mathcal{W}} \sum_{k \in \mathcal{K}_w: k \ni a} d_w(\mathbf{p}_w^*)_k - f_a \right)_{a \in \mathcal{A}}$$

is a descent direction. This leads to a descent algorithm for solving  $\min_{\mathbf{f}} \Phi_f(\mathbf{f})$ , which also calculates equilibrium flows for the CMM-SUE formulation.

**Input:** Parameters: Directed graph  $\mathcal{G} = (\mathcal{N}, \mathcal{A})$ ; set of origin-destination pairs  $\mathcal{W}$ ; demand  $d_w$ , set of paths  $\mathcal{K}_w$ , and covariance matrix  $\Sigma_w, \forall w \in \mathcal{W}$ ; cost function  $c_a(f_a), \forall a \in \mathcal{A}$ ; starting point:  $\mathbf{f}^0 = \mathbf{0}$ ; tolerance:  $\bar{\epsilon}$ .

**Output:** Equilibrium arc flow vector  $\mathbf{f}^{(i)}$  for CMM-SUE.

$i \leftarrow 0$ ,

**while** the stopping criterion is violated, **do**

	Obtain $\mathbf{p}_w^{(i)*}$ from Algorithm 1 with $\mathbf{c}_w$ and $\Sigma_w, \forall w \in \mathcal{W}$ .
	$\mathbf{y}_a^{(i)} = \sum_{w \in \mathcal{W}} \sum_{k \in \mathcal{K}_w: k \ni a} d_w(\mathbf{p}_w^{(i)*})_k, \forall a \in \mathcal{A}$ .
	$\mathbf{f}^{(i+1)} \leftarrow \mathbf{f}^{(i)} + \frac{1}{i}(\mathbf{y}^{(i)} - \mathbf{f}^{(i)})$ .
	$i \leftarrow i + 1$

**end**

**Algorithm 2:** Algorithm for CMM-SUE.

Since the optimality condition is  $\sum_{w \in \mathcal{W}} \sum_{k \in \mathcal{K}_w: k \ni a} d_w(\mathbf{p}_w^*)_k - f_a = 0, \forall a \in \mathcal{A}$ , in Algorithm 2 we use the stopping criterion

$$\frac{\|\mathbf{y}^{(i)} - \mathbf{f}^{(i)}\|}{\|\mathbf{f}^{(i)}\|} \leq \bar{\epsilon}$$

to terminate the algorithm. Algorithm 2 is similar to the traditional MSA algorithm with the main difference in the calculation of the choice probabilities  $\mathbf{p}_w^*$  at each iteration of the algorithm. In the CMM-SUE model, we use the projected gradient ascent algorithm at each iteration to compute choice probabilities. In terms of implementation this can be used to provide computational benefits for finding equilibrium flows. For example to initialize Algorithm 1 at each iteration  $i$ , we can make use of the optimal choice probability vector  $\mathbf{p}_w^{(i-1)*}$  from the previous iteration. This provides faster convergence in the inner algorithm. As we will demonstrate in the next section, Algorithm 2 outperforms MSA for MNP-SUE in terms of computational efficiency.

## 6 Numerical Tests

In this section, we compare the MNP-SUE and CMM-SUE for five transportation networks discussed in the literature. The first example is a two-link road network for which the CMM-SUE arc flow can be found explicitly. The second example is a five-link road network taken from Daganzo [23] with linear cost functions while the third example is the classical Braess paradox network taken

from Prashker and Bekhor [42] with nonlinear cost functions. The first three examples have a single origin-destination pair. The fourth example is a grid graph with nonlinear cost function and multiple origin-destination pairs while the last example is a larger Sioux Falls network consisting of 24 nodes and 76 links. We compare the results of the MNP-SUE and CMM-SUE models in terms of the equilibrium flows and the computational times required to calculate these. All the codes were run in MATLAB<sup>2</sup>.

## 6.1 A Simple Example with Two Links

Consider the simple two link network discussed in Section 3.1. Assume that 1 unit of traffic flows from node  $A$  to  $B$  with a flow of  $x$  on link 1 and a flow of  $1 - x$  on link 2. Travelers perceive the utility of link 1 as:

$$\tilde{u}_{11}(x) = -c_1^{(0)} - c_1^{(1)}x + \tilde{\epsilon}_{11},$$

and the utility of link 2 as:

$$\tilde{u}_{21}(1 - x) = -c_2^{(0)} - c_2^{(1)}(1 - x) + \tilde{\epsilon}_{21},$$

where  $c_1^{(1)}, c_2^{(1)} \geq 0$  to reflect that congestion increases costs. The link costs are affine functions of the flow value in this example. Assume that  $(\tilde{\epsilon}_1, \tilde{\epsilon}_2)^T$  follows a multivariate normal distribution with  $\sigma^2 = \text{Var}_{\theta_1}[\tilde{\epsilon}_{11}] = \text{Var}_{\theta_1}[\tilde{\epsilon}_{21}]$  and  $\rho = \text{Corr}_{\theta_1}[\tilde{\epsilon}_{11}, \tilde{\epsilon}_{21}]$ . Then the equilibrium arc flow for the MNP-SUE from Section 3.1 is given by the solution to the equation:

$$x^{\text{mnp}} = \Phi(-z_1 x^{\text{mnp}} + z_2), \quad (6.25)$$

where

$$z_1 = \frac{c_1^{(1)} + c_2^{(1)}}{\sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}} \geq 0 \text{ and } z_2 = \frac{-c_1^{(0)} + c_2^{(0)} + c_2^{(1)}}{\sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}},$$

and  $x^{\text{mnp}}$  denotes the equilibrium arc flow on link 1. While it is not possible to derive an explicit expression for  $x^{\text{mnp}}$ , the solution can be found by using a simple bisection search method. On the other hand, without the normality assumption, the arc flow in the CMM-SUE model is given as the solution to the equation (see Section 3.1 and Formulation (4.19)):

$$x^{\text{cmm}} = \frac{1}{2} \left( 1 + \frac{-z_1 x^{\text{cmm}} + z_2}{\sqrt{(-z_1 x^{\text{cmm}} + z_2)^2 + 1}} \right), \quad (6.26)$$

where  $x^{\text{cmm}}$  denotes the equilibrium arc flow on link 1. Equivalently  $x^{\text{cmm}}$  is the root to the quartic equation:

$$4z_1^2 x^4 - (4z_1^2 - 8z_1 z_2) x^3 + (4z_2^2 + 4 + 8z_1 z_2 - z_1^2) x^2 - (4 + 4z_2^2) x + 1 = 0. \quad (6.27)$$

While it is possible to provide an explicit algebraic expression for  $x^{\text{cmm}}$  in terms of the parameters of the problem, the formula is extremely unwieldy. Note that quartic equations are the highest degree polynomials for which every polynomial equation has a closed-form algebraic solution. We compare the equilibrium flows for link 1 from the CMM and MNP models in Figure 6.3 for different values of the parameters  $z_1$  and  $z_2$ . Figure 6.3 plots the absolute difference in the equilibrium flows  $|x^{\text{cmm}} - x^{\text{mnp}}|$ . The flows are quite close to each other in all the cases, with the maximum difference of around 0.0321 (3.21%). Note that this result indicates that for the two link case with affine cost functions, the CMM and the MNP models have the largest difference when the costs are flow independent (see Section 3.1) with the equilibrium flow values being typically much closer.

<sup>2</sup>The MATLAB code for the CMM-SUE model and the examples discussed in this paper are available at the website of the fourth author [http://people.sutd.edu.sg/~natarajan.karthik/?page\\_id=2](http://people.sutd.edu.sg/~natarajan.karthik/?page_id=2).



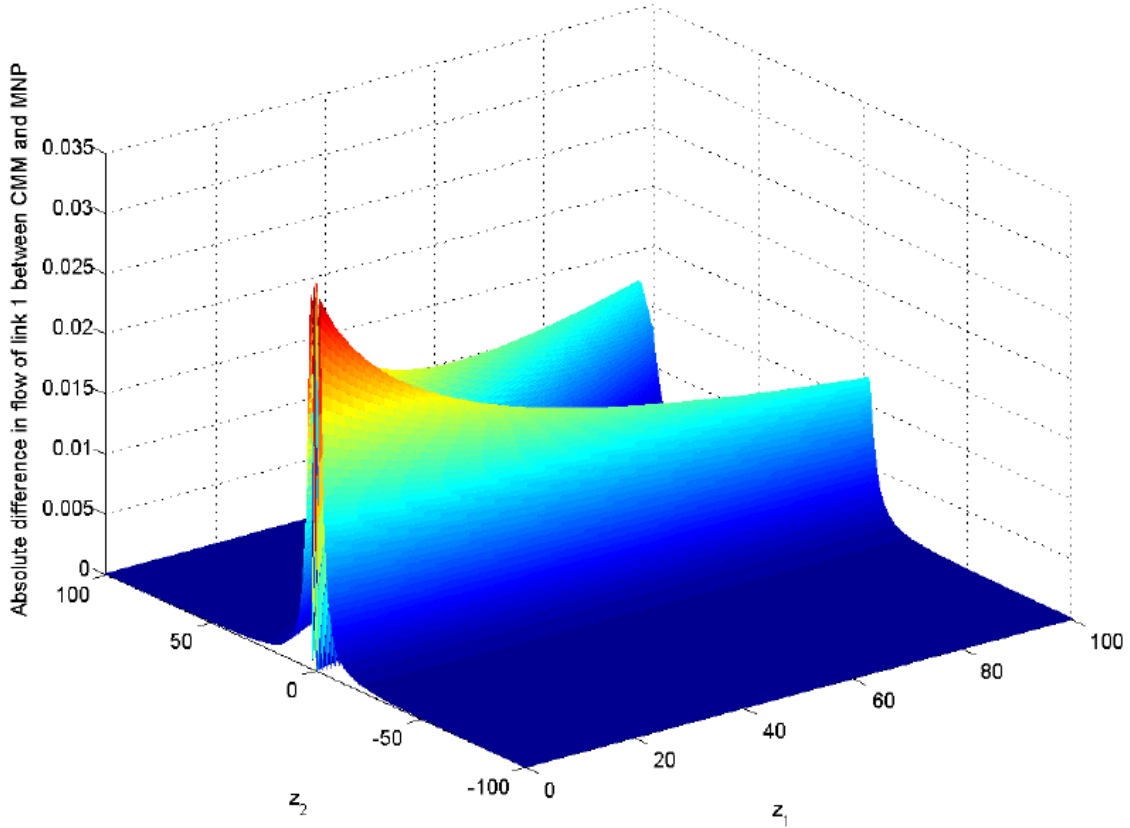


Figure 6.3: Absolute difference in flow of link 1 in the two link network as a function of  $z_1$  and  $z_2$ .

## 6.2 Five link, three path example from Daganzo [23]

The next example is a small network (Figure 6.4) taken from Daganzo [23]. This network has 5 links, 3 paths, and 100 units of flow between a single origin-destination pair. The deterministic

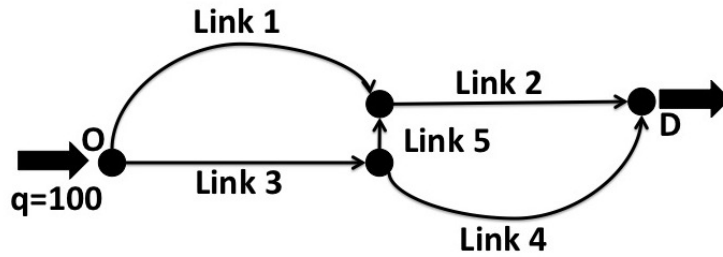


Figure 6.4: Route choice network with three alternative paths.

costs associated with links are linear functions of the link flows:

$$c_1(f_1) = 7 + \frac{f_1}{22}, \quad c_2(f_2) = 5 + \frac{f_2}{78}, \quad c_3(f_3) = 5 + \frac{f_3}{78}, \quad c_4(f_4) = 7 + \frac{f_4}{22}, \quad c_5(f_5) = \frac{f_5}{56}.$$

The random component associated with each link is assumed to have zero mean and unit variance. It is assumed that these random variables are independent for each link.

Given a single origin-destination pair, we denote the set of paths as  $\mathcal{K} = \mathcal{K}_1 = \{1, 2, 3\}$ , where path 1 corresponds to links 1 and 2, path 2 corresponds to links 3 and links 4, and path 3 corresponds to links 3, 5 and 2. Given a feasible arc flow vector  $\mathbf{f} = (f_1, f_2, \dots, f_5)^T$ , the choice probabilities of the paths can be calculated from a choice model (either CMM or MNP) with mean cost  $\mathbf{c} = (c_1(f_1) + c_2(f_2), c_3(f_3) + c_4(f_4), c_3(f_3) + c_5(f_5) + c_2(f_2))^T$  and covariance matrix:

$$\Sigma_1 = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 3 \end{pmatrix}.$$

Daganzo [22] showed that the equilibrium link flow for the MNP-SUE model is:

$$\mathbf{f}^{\text{mnp}} = (22, 78, 78, 22, 56)^T$$

and the corresponding equilibrium link cost is:

$$\mathbf{c}^{\text{mnp}} = (8, 6, 6, 8, 1)^T.$$

A table providing the flow values of the first 100 iterations of the MSA algorithm (which converges to MNP-SUE flows) is provided in [22]. It is observed that the flow values change rapidly in the beginning, but the convergence slows down after a while. After 100 iterations the flows are within 20 percent of the optimal values. We use Algorithm 2 to calculate the CMM-SUE and present the results in Table 6.1 and 6.2. Table 6.1 provides the link flows, link costs, and the choice probabilities of paths after 40 iterations of the algorithm in the second column. We provide the MNP-SUE link flows, link costs, and path probabilities in the third column for comparison. We also provide the flow values over the 40 iterations of Algorithm 2 in more detail in the Table 6.2. The last row of this table also includes the flow values after 100 iterations of the MSA algorithm. The computational time needed for 40 iterations of Algorithm 2 is 0.46 seconds and for 100 iterations 0.66 seconds; on the other hand the MSA algorithm for the MNP model takes 6.3 seconds to perform 40 iterations and 13.5 seconds for 100 iterations. Interestingly, the CMM-SUE flows after a smaller number of iterations is closer to the exact MNP-SUE flows than those obtained from the MSA algorithm. Even though we start the two algorithms with the same initial flows, Algorithm 2 finds flows which are within 20 percent of the exact MNP-SUE flows in just a few iterations. This is a result of the slow convergence of the MSA algorithm for MNP-SUE and the fact that CMM-SUE and MNP-SUE are relatively close to each other. We also observe that the CMM-SUE flows do not change much even if we let the algorithm run for a long time.

Table 6.1: Comparison of results between CMM and MNP model after 40 iterations.

Variables	CMM-SUE (approximate)	MNP-SUE (exact)
Link flow	(21.56, 78.44, 78.44, 21.56, 56.88)	(22, 78, 78, 22, 56)
Link cost	(7.980, 6.005, 6.005, 7.980, 1.015)	(8, 6, 6, 8, 1)
Path probability	(0.215, 0.215, 0.568)	(0.22, 0.22, 0.56)
Total travel cost	1344	1344

Given a feasible flow vector  $\mathbf{f}$ , the total travel cost is calculated as

$$T(\mathbf{f}) = \sum_{a \in \mathcal{A}} f_a c_a(f_a).$$

The system optimal (SO) solution is a flow vector,  $\mathbf{f}^{\text{so}}$ , which minimizes the  $T(\mathbf{f})$ . The final row in Table 6.1 provides the total travel cost associated with the CMM-SUE and MNP-SUE flows. The travel costs are (almost for all practical purposes) equal for both the equilibriums.

Table 6.2: Link flows in CMM and MNP model.

Iteration	Link 1	Link 2	Link 3	Link 4	Link 5
1	0	0	100	100	0
10	21.5277	78.2810	78.4723	21.7190	56.7532
20	21.5553	78.4089	78.4447	21.5911	56.8536
30	21.5596	78.4262	78.4404	21.5738	56.8666
35	21.5605	78.4295	78.4395	21.5705	56.8690
36	21.5606	78.4300	78.4394	21.5700	56.8693
37	21.5608	78.4304	78.4392	21.5696	56.8696
38	21.5609	78.4308	78.4391	21.5692	56.8699
39	21.5610	78.4312	78.4390	21.5688	56.8702
40	21.5611	78.4315	78.4389	21.5685	56.8704
Daganzo [23] (After 100 iterations)	21	79	79	21	58
Theoretical MNP	22	78	78	22	56

### 6.3 Braess Paradox Example

The Braess paradox occurs when adding extra capacity to a network in which drivers choose their routes selfishly, can cause an increase in the overall travel costs. This is because the Nash equilibrium of such a system is not necessarily optimal for the whole system. It is known that Braess paradox occurs with both linear and nonlinear cost functions for certain ranges of demand [see Prashker and Bekhor [8]].

In this section, we consider a simple network (given in Figure 6.5) with four nodes, five links, three paths, and a single origin destination pair (nodes A and B). The link costs are nonlinear functions of the link flows (as in [8]) as follows:

$$c_1(f_1) = 1 + 2(f_1/s_1)^2, \quad c_2(f_2) = 10 + (f_2/s_2)^2, \quad c_3(f_3) = 1 + 2(f_3/s_3)^2,$$

$$c_4(f_4) = 10 + (f_4/s_4)^2, \quad c_5(f_5) = 1 + 2(f_5/s_5)^2.$$

The parameter  $s_i$  captures the capacity of link  $i$ , which is set to  $s_i = 3.2$  for all  $i$ . Our goal is to compare the total travel cost associated to the equilibrium flows with respect to the total travel cost of the system optimum flows. We consider the deterministic user equilibrium (UE), variants of the MNL-SUE with different parameters, the MNP-SUE, and the CMM-SUE. In addition, we also study the effect of removing Link 3 on the total travel cost under each equilibrium.

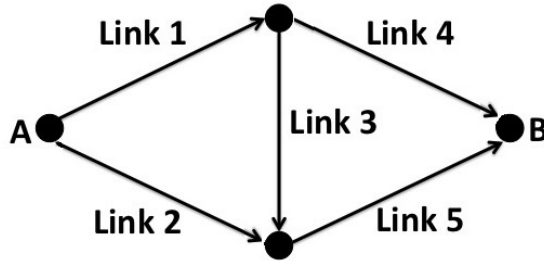


Figure 6.5: Braess Paradox Network.

First consider the 4-link network, where link 3 is removed. In this case, there are only two paths with symmetric travel costs. No matter how large the demand is, it would be distributed

equally between the two paths in any (deterministic or stochastic) equilibrium, which is also the system optimal solution. We capture the change in the total travel cost due to the existence of link 3 under various equilibria in Figure 6.6. The vertical axis provides the deviation of the total travel cost corresponding to different equilibrium flows from the optimal total travel cost on the network without link 3. The total traffic flow in the network (total demand between the origin and the destination) is given on the horizontal axis. We observe that when the demand is low adding the extra link is beneficial. As the demand increases the Braess paradox is observed (for different ranges of demand) for all equilibrium solutions, i.e., the total travel cost with the extra link becomes higher than without it. The CMM-SUE and the MNP-SUE equilibria are quite close to each other, and they are more robust to the paradox than the UE equilibrium when the demand is mild. When the network is very congested, i.e., the demand is high, the deterministic UE performs slightly better.

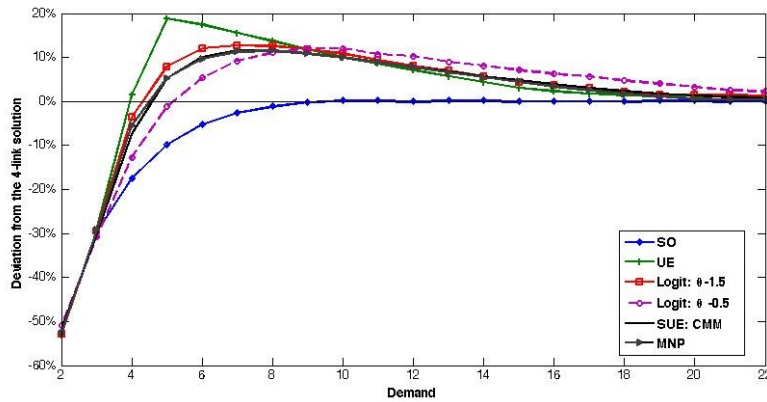


Figure 6.6: Deviation from the 4-link solution.

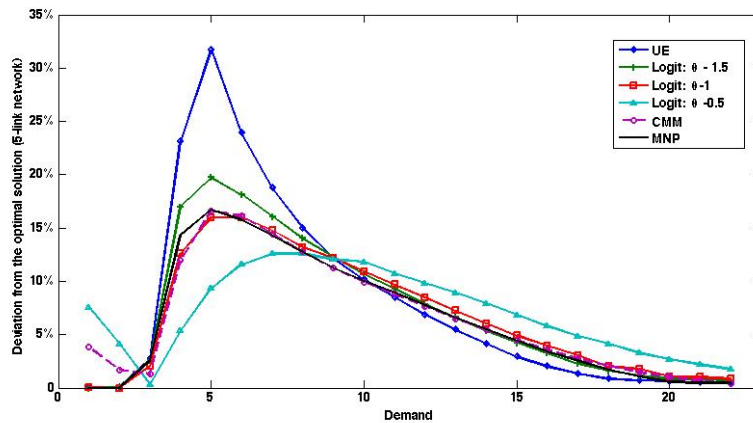


Figure 6.7: Deviation from the optimal solution (5-link network).

We also compare the deterministic UE and SUE solutions with the SO solution on the original network. We measure the deviation of the total travel cost relative to the system optimal solution as:

$$\frac{T(\mathbf{f}) - T(\mathbf{f}^{\text{so}})}{T(\mathbf{f}^{\text{so}})},$$

where  $\mathbf{f}$  denotes the equilibrium flow we are referring to and  $\mathbf{f}^{\text{so}}$  denotes the system optimal flow. These results can be seen in Figure 6.7 again as a function of demand. The inflection point at 9.6

demand units suggests that the congestion effect is large enough to bring the UE and SO solutions closer to each other. For moderate congestion, the travel costs associated with all SUE flows are closer to the SO travel costs than the cost of the deterministic UE flows. The worst UE performance case is around 5 demand units with the highest deviation from optimal solution. The CMM-SUE flows are at par and at certain times outperform the ones calculated through either MNL or MNP models.

## 6.4 Grid Network

In this section, we consider the grid network in Figure 6.8 which consists of 12 nodes, 34 links and 132 origin-destination pairs. The paths in the choice set for any origin-destination pair is the set of all paths with the minimum number of links between the nodes. For example, there is only one possible path from node 1 to node 4 which is path 1 – 3 – 5, and there are  $\binom{5}{2} = 10$  possible paths from node 1 to node 12. The link cost functions are given as:

$$c_a = \bar{c}_a \left( 1 + B \left( \frac{f_a}{s_a} \right)^t \right). \quad (6.28)$$

The purpose of the numerical test is to compare the total travel costs, equilibrium flows, and computational times of the CMM-SUE and MNP-SUE models. The test data is generated as follows:

- (i) For every origin-destination pair, the demand  $d_w$  is an integer randomly generated from [100, 1000].
- (ii) For every link, the free flow cost  $\bar{c}_a$  is randomly generated from a uniform distribution  $U(10, 20)$ . The capacity  $s_a = 0.5 \sum_{w \in \mathcal{W}} d_w \cdot r$  where  $r$  is randomly generated by a uniform distribution  $U(1, 2)$  and  $B$  is set to be 0.2. We consider two cases of the parameter  $t$ ,  $t = 1$  and  $t = 4$ .
- (iii) For every origin-destination pair, the covariance matrices for the random error term of the path cost are randomly generated positive definite matrices whose diagonals are the number of links on the path.

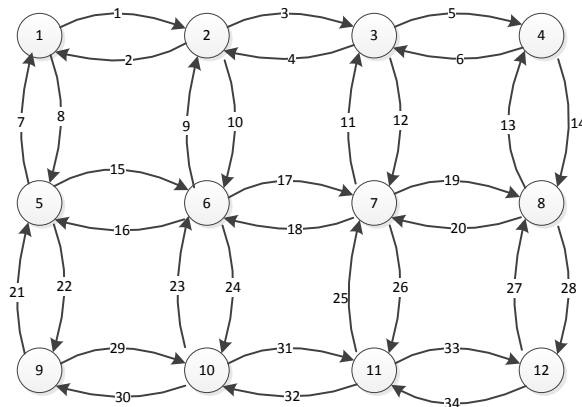


Figure 6.8: Grid Network.

We compute the equilibrium flows of CMM-SUE and MNP-SUE for the grid network with all the 132 origin-destination pairs and all the possible paths. The stopping criteria for the outer

iteration (Algorithm 2) is set to be  $10^{-4}$ . For CMM-SUE, Algorithm 1 is used to compute the choice probability  $\mathbf{p}_w$  for the inner iteration. The stopping criteria of the inner iteration is set to be  $10^{-3}$ , and the step size is obtained by the backtracking line search. The starting point in the inner algorithm is set to be the optimal choice probability from the previous iteration, which makes the algorithm more efficient since the optimal choice probability for the two inner iterations are close especially when the choice probabilities are close to the CMM-SUE optimal choice probability. For MNP-SUE, the choice probability  $\mathbf{p}_w$  is obtained by the Monte Carlo simulation with  $10^5$  samples. In Table 6.3 and Table 6.4 we report the computational time (in seconds) and iteration numbers for 10 scenarios of the data with  $t = 1$  and  $t = 4$ , respectively. We also report the differences between the total travel cost associated with the equilibrium flows (relative to the total cost of the MNP-SUE flow) in the last column. The, so-called, relative difference is calculated as:

$$\frac{T(\mathbf{f}^{\text{cmm}}) - T(\mathbf{f}^{\text{mnp}})}{T(\mathbf{f}^{\text{mnp}})},$$

where  $\mathbf{f}^{\text{cmm}}$  and  $\mathbf{f}^{\text{mnp}}$  are the CMM-SUE and MNP-SUE flows, respectively. The equilibrium flows of the two models are shown in Figure 6.9. We observe that the relative difference of the total cost is very close to 0, which means that the total cost of CMM-SUE and MNP-SUE are almost the same. Moreover, from Figure 6.9 we see that the CMM-SUE flows and MNP-SUE flows are very close. The computational time required to calculate the CMM-SUE flows is much smaller than that of calculating the MNP-SUE flows. Hence we can conclude that for this network, CMM-SUE dominates MNP-SUE since the optimal solutions for the two models are approximately the same while CMM-SUE has a significant advantage in the computational time.

Table 6.3: Computational times and iteration numbers ( $t = 1$ ).

Samples	CMM-SUE		MNP-SUE		Relative difference
	time	iterations	time	iterations	
1	13.99	102	39.88	120	4.80e-005
2	8.45	109	40.07	129	6.03e-005
3	10.20	81	28.18	89	5.48e-005
4	7.46	90	35.40	106	5.98e-005
5	8.03	134	57.14	163	3.79e-005
6	11.19	128	47.62	146	8.58e-005
7	9.49	109	42.18	126	5.46e-005
8	5.74	72	25.24	77	5.42e-005
9	9.10	100	37.81	117	4.38e-005
10	12.69	111	36.59	115	8.18e-005

Table 6.4: Computational times and iteration numbers ( $t = 4$ ).

Samples	CMM-SUE		MNP-SUE		Relative difference
	time	iterations	time	iterations	
1	1.34	5	2.84	6	-6.95e-006
2	2.84	10	4.19	11	-1.04e-005
3	2.39	18	6.26	16	-1.24e-005
4	1.11	2	1.81	3	-2.37e-006
5	2.14	19	7.10	19	-2.19e-005
6	0.95	3	2.27	4	-2.64e-006
7	1.94	6	3.21	7	-6.39e-006
8	1.67	3	1.87	3	-2.66e-006
9	1.28	3	1.87	3	-2.08e-006
10	1.80	5	2.83	6	-3.20e-006

## 6.5 Sioux Falls Network

In this section, we focus on a real transportation network, which is much larger than our previous examples. The aggregated network of the city of Sioux Falls, South Dakota, displayed in Figure

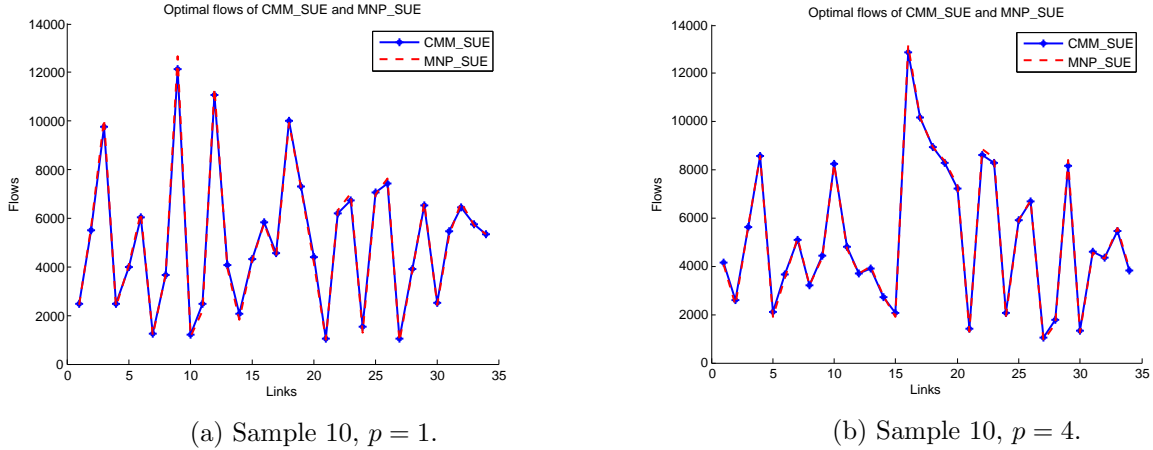


Figure 6.9: CMM-SUE and MNP-SUE flows.

6.10, consists of 24 nodes, 76 links, and 552 origin-destination pairs. This network has been extensively used in the traffic equilibrium literature [see Morlok et al. [37], LeBlanc [30], LeBlanc, Morlok and Pierskella [32], Leblanc and Morlok [31] and Abdulaal and Leblanc [2]]. The link cost functions have the same expression as in (6.28). The related data such as link parameters of the cost functions and the demand between origin-destination pairs, collected by the Bureau of Public Research, can be found in Morlok et al. [37].

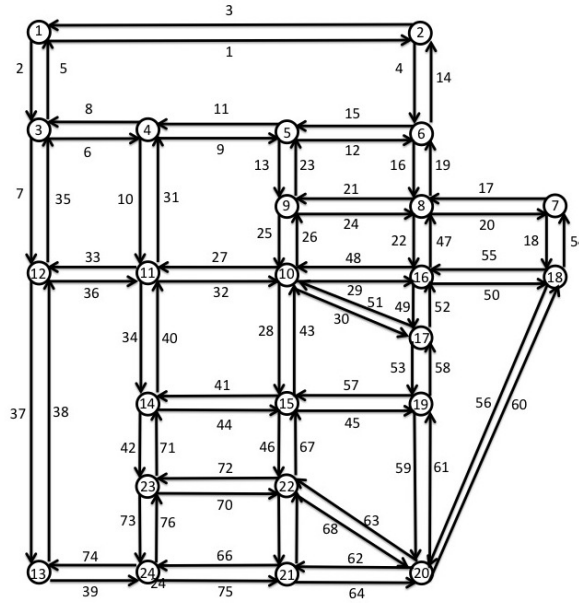


Figure 6.10: Sioux Falls Network.

We compute the CMM-SUE and MNP-SUE flows on the network by changing the numbers of origin-destination pairs and the number of paths between the origin-destination pairs. The number of origin-destination pairs is varied between 10 to 552 and the number of paths is varied between 3 and 10 for every origin-destination pair. The computational time (in seconds) and relative difference between the total travel costs of the two equilibrium flows are reported in Table 6.5. The stopping criteria for the outer iterations is set to be  $10^{-3}$  and the other algorithm parameters (starting points, inner stopping criteria, and the sample size for the Monte Carlo simulation) is set to be the

same as in the grid network (Section 4.4) example.

Table 6.5: Computational times and relative difference of total costs of the CMM-SUE and MNP-SUE flows.

OD pairs	No. of paths	3 paths	4 paths	5 paths	6 paths	7 paths	8 paths	9 paths	10 paths
10	time CMM	0.07	0.10	0.12	0.15	0.17	0.23	0.27	0.30
	time MNP	0.16	0.20	0.26	0.32	0.36	0.40	0.46	0.50
	time ratio	2.11	2.08	2.21	2.16	2.13	1.76	1.75	1.69
	cost difference	4.80e-008	6.19e-008	9.25e-008	9.51e-008	1.31e-007	1.78e-007	2.29e-007	2.24e-007
50	time CMM	0.30	0.59	0.54	0.64	0.78	0.93	1.06	1.31
	time MNP	0.81	1.02	1.29	1.53	1.83	2.07	2.54	2.74
	time ratio	2.68	1.73	2.40	2.39	2.34	2.23	2.41	2.09
	cost difference	8.25e-006	6.86e-006	7.75e-006	5.09e-006	6.51e-006	8.92e-006	1.01e-005	9.87e-006
100	time CMM	0.75	1.01	1.29	1.65	2.04	2.71	3.44	4.28
	time MNP	2.30	3.05	3.11	5.92	6.59	8.42	10.05	11.03
	time ratio	3.04	3.02	2.42	3.59	3.23	3.11	2.92	2.57
	cost difference	9.13e-006	1.38e-005	1.14e-005	1.75e-005	2.82e-005	3.49e-005	3.64e-005	4.19e-005
200	time CMM	3.48	6.06	8.50	11.02	16.48	21.07	26.34	30.98
	time MNP	20.61	34.61	44.71	61.80	81.30	94.36	116.90	131.32
	time ratio	5.92	5.71	5.26	5.61	4.93	4.48	4.44	4.24
	cost difference	1.78e-004	3.24e-004	4.44e-004	5.16e-004	6.11e-004	6.75e-004	7.36e-004	7.79e-004
300	time CMM	11.57	18.45	27.81	35.96	49.09	58.95	76.97	94.43
	time MNP	80.61	128.29	194.47	246.06	290.02	318.42	437.99	467.31
	time ratio	6.97	6.95	6.99	6.84	5.91	5.40	5.69	4.95
	cost difference	1.13e-003	2.12e-003	3.10e-003	3.71e-003	4.42e-003	4.97e-003	5.38e-003	5.92e-003
400	time CMM	18.88	34.25	52.31	73.87	104.61	129.41	172.64	210.12
	time MNP	136.16	262.13	367.83	506.45	610.91	781.89	957.77	1204.84
	time ratio	7.21	7.65	7.03	6.86	5.84	6.04	5.55	5.73
	cost difference	2.58e-003	4.88e-003	6.96e-003	8.60e-003	1.03e-002	1.19e-002	1.31e-002	1.45e-002
552	time CMM	26.58	47.82	84.11	119.44	156.10	210.10	286.46	358.09
	time MNP	186.64	342.60	483.98	685.20	881.98	1250.48	1535.29	1930.32
	time ratio	7.02	7.16	5.75	5.74	5.65	5.95	5.36	5.39
	cost difference	5.16e-003	9.41e-003	1.43e-002	1.79e-002	2.16e-002	2.51e-002	2.81e-002	3.15e-002

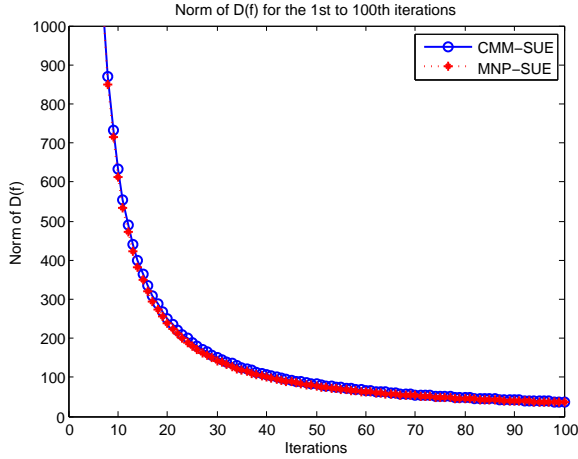
From Table 6.5, we observe that the relative difference of the total cost for CMM-SUE and CMM-MNP is very small, but the computational time required to calculate the CMM-SUE flows is an order of magnitude smaller than that of calculating the MNP-SUE flows in all the cases. In the table, the “time ratio” is defined as  $\text{time ratio} = \frac{\text{time MNP}}{\text{time CMM}}$ . We observe that as the number of OD pairs increases, the relative advantage of the CMM-SUE in computational times becomes more prominent. On the other hand, the relative advantage as the number of paths increases decreases slightly. It is clear however that the CMM model is more efficiently computable. For both CMM-SUE and MNP-SUE, the equilibrium flow is the solution to the equation:

$$D(\mathbf{f})_a := f_a - \sum_{w \in \mathcal{W}} d_w \sum_{k \in \mathcal{K}_w: k \ni a} \mathbb{P} \left( k = \arg \max_{l \in \mathcal{K}_w} (-c_l(\mathbf{f}) + \tilde{\epsilon}_l) \right) = 0, \quad \forall a \in \mathcal{A}. \quad (6.29)$$

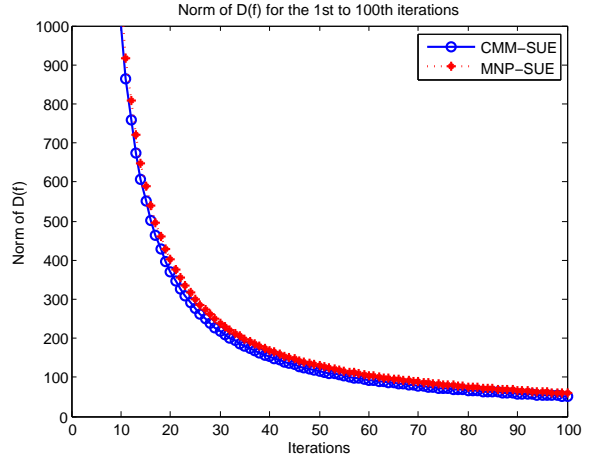
We show the result for the norm of  $D(\mathbf{f})$  with 552 OD pairs in Figure 6.11 for the 1st to 100th iterations of Algorithm 2. The figure indicates that the algorithm converges rapidly early on and then slows down as you get closer to optimality. This is a standard feature of first order optimization methods.

We also provide the CMM-SUE and MNP-SUE flows for 552 OD pairs with 3 paths on every OD pair using two figures. Figure 6.12 provides two graphs, corresponding to the two different equilibriums, where each link has width proportional to the flow on it. It can be seen that the link flow values of the two models are fairly similar. Actual values on all 76 links are provided in Figure 6.13 for a more refined comparison. This concludes our last test, as we have shown that both the total travel times and the equilibrium flows are fairly similar for both equilibrium models.



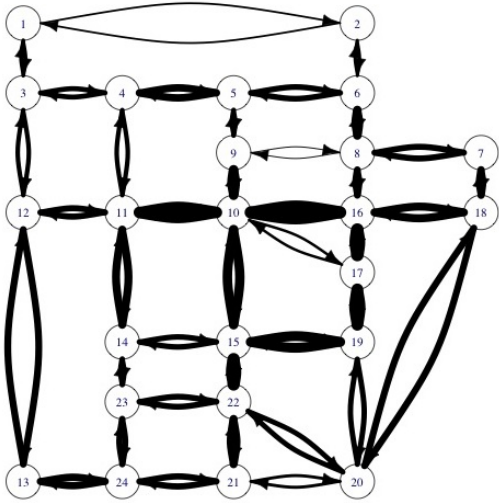


(a) 3 paths for each OD pair.

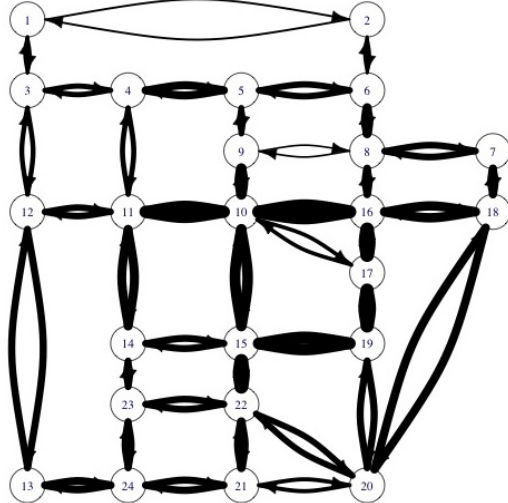


(b) 5 paths for each OD pair.

Figure 6.11: Norm of  $D(f)$  for the 1st to 100th iterations.



(a) MNP-SUE



(b) CMM-SUE

Figure 6.12: The CMM-SUE and MNP-SUE flows.

## 7 Conclusion

In this paper, we introduce a new stochastic user equilibrium that models the effect of congestion on the distribution of the traffic on the links of a network. The underlying route choice model is based on a recent distributionally robust discrete choice model that is considered in the transportation literature for the first time to the best of our knowledge. The Cross Moment model uses the mean and covariance information for the utilities of different paths without any further distributional assumptions. Therefore, it is richer than the logit based equilibrium models and robust to distributional assumptions in comparison to the probit based equilibrium model.

We show that the equilibrium flows are equal to the first order conditions of a minmax optimization problem. We propose a simple gradient descent algorithm to calculate the equilibrium

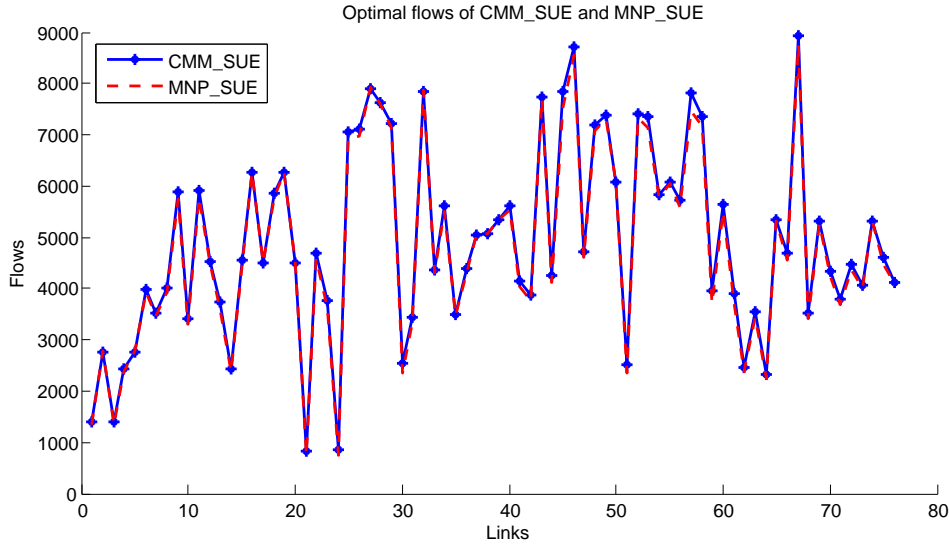


Figure 6.13: CMM-SUE and MNP-SUE flows when there are 552 OD pairs.

flows and demonstrate that it is capable of solving medium sized networks efficiently. Using this method, we demonstrate on some small artificial networks as well as a larger network that the new equilibrium results in flows similar to those from the probit based model and outperforms the deterministic user equilibrium when tested on a network with respect to the Braess paradox. In addition, the descent algorithm converges much faster than similar methods for probit based models and is robust to the choice of parameters of the algorithm such as the stepsizes.

We believe that the rich and robust nature of the model together with its computational tractability makes the proposed distributionally robust equilibrium interesting for further study, especially on larger scale networks with added features such as multiple vehicle types and elastic demands.

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## A Appendix: Additional Notation

$\mathcal{G} = (\mathcal{N}, \mathcal{A})$	Directed graph with set of nodes $\mathcal{N}$ and set of arcs/links $\mathcal{A}$
$\mathcal{W}$	Set of origin-destination (OD) pairs in $\mathcal{G}$
$(r_w, s_w)$	The $w$ th OD pair with the origin $r_w \in \mathcal{N}$ with the destination $s_w \in \mathcal{N}$
$\mathcal{K}_w$	Directed simple paths between $r_w$ and $s_w$
$\mathcal{K}$	Set of all simple paths in $\mathcal{G}$ , i.e., $\bigcup_{w \in \mathcal{W}} \mathcal{K}_w$
$d_w$	Demand associated with the $w$ th OD pair
$\mathbf{x} = (x_{kw})_{k \in \mathcal{K}_w, w \in \mathcal{W}}$	The path flow vector, where $x_{kw}$ denotes the amount of flow on path $k \in \mathcal{K}_w$
$\mathbf{f} = (f_a)_{a \in \mathcal{A}}$	The arc flow vector, where $f_a$ is the flow on arc $a \in \mathcal{A}$
$c_a(f_a)$	The deterministic cost of arc $a \in \mathcal{A}$ with $f_a$ units of flow <sup>3</sup>
$\underline{c}_a(f_a)$	Cost of arc $a \in \mathcal{A}$ with zero arc flow
$\tilde{\mathbf{u}} = (\tilde{u}_{kw})_{k \in \mathcal{K}_w, w \in \mathcal{W}}$	The total path utility vector, where $\tilde{u}_{kw}$ correspond to $k \in \mathcal{K}_w$
$\mathbf{c}(\mathbf{f}) = (c_{kw}(\mathbf{f}))_{k \in \mathcal{K}_w, w \in \mathcal{W}}$	The path cost vector, where $c_{kw}(\mathbf{f})$ is the deterministic cost on $k \in \mathcal{K}_w$
$\tilde{\boldsymbol{\epsilon}} = (\tilde{\epsilon}_{kw})_{k \in \mathcal{K}_w, w \in \mathcal{W}}$	The random comp. of the total utility vector, where $\tilde{\epsilon}_{kw}$ correspond to $k \in \mathcal{K}_w$
$\mathbf{p}(\mathbf{f}) = (p_{kw}(\mathbf{f}))_{k \in \mathcal{K}_w, w \in \mathcal{W}}$	The choice probability vector, where $p_{kw}$ is the prob. that $k \in \mathcal{K}_w$ is chosen
$\mathbf{x}_w(\mathbf{f}) = (x_{kw}(\mathbf{f}))_{k \in \mathcal{K}_w}$	The $w$ th path flow vector, i.e., subvector of $\mathbf{x}(\mathbf{f})$ corr. to the $w$ th OD pair
$\mathbf{c}_w(\mathbf{f}) = (c_{kw}(\mathbf{f}))_{k \in \mathcal{K}_w}$	The $w$ th path cost vector, i.e., subvector of $\mathbf{c}(\mathbf{f})$ corr. to the $w$ th OD pair
$\tilde{\boldsymbol{\epsilon}}_w = (\tilde{\epsilon}_{kw})_{k \in \mathcal{K}_w}$	Subvector of $\tilde{\boldsymbol{\epsilon}}$ corr. to the $w$ th OD pair
$\mathbf{p}_w(\mathbf{f}) = (p_{kw}(\mathbf{f}))_{k \in \mathcal{K}_w}$	The $w$ th choice probability vector, i.e., subvector of $\mathbf{p}(\mathbf{f})$ corr. to the $w$ th OD pair
$\theta_w$	Distribution of the random vector $\tilde{\boldsymbol{\epsilon}}_w$
$\Theta_w$	A given set of distribution functions to which $\theta_w$ belongs
$\boldsymbol{\Sigma}_w$	Covariance matrix of random vector $\tilde{\boldsymbol{\epsilon}}_w$
$T(\mathbf{f})$	Total mean travel cost associated with the total arc flow vector $\mathbf{f}$
$\mathcal{S}_w$	The $ \mathcal{K}_w $ -dimensional unit simplex
$\mathcal{S}$	The product of simplices $\mathcal{S}_1 \times \dots \times \mathcal{S}_{ \mathcal{W} }$

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