New Analytical Bounds on the Average Undershoot in an Infinite Horizon (s,S) Inventory System

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Abstract

We provide new bounds on the average undershoot in an infinite horizon (s, S) inventory system with i.i.d demands with known moments up to order three. The two moment bound improves on Lorden's bound (Ann. Math. Stat. 1970) for small values of $S - s$ and low values of the coefficient of variation. The three moment bound provides further improvement for negatively skewed demands. The bounds are sharp and shown to be useful when the asymptotically tight two moment approximation fails.

Keyword: Inventory; Moment bounds; Semidefinite program

1 Introduction

Consider a single product periodic review inventory system with random demands. We restrict our attention to the simplest setting with zero lead times and backlogging of all unmet demand. The demand across different periods are independent and identically distributed. Let D_n represent the demand in period n for $n = 0, 1, 2, \ldots$ and D represent the generic demand random variable. Under the (s, S) order policy, the order amount in each period is based on two numbers s and S where $0 \leq s < S$. At the start of each period, if the inventory level is strictly below s, a positive order is placed to bring it up to S, otherwise no order is made. The optimality of the (s, S) policy has been established for particular cost models by Scarf [19] and Iglehart [10]. Let W_n be the inventory level at the beginning of period n with $W_0 = 0$. The recursive equation governing the inventory level is given by:

$$W_{n+1} = \begin{cases} S - D_n & \text{if } W_n < s \\ W_n - D_n & \text{if } W_n \geq s. \end{cases} \tag{1}$$

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In the steady state version of the model, W_n converges in distribution to a random variable W as $n \to \infty$ (see Karlin [11]). The equation describing W is:

$$W =_d \begin{cases} S - D & \text{if } W < s \\ W - D & \text{if } s \leq W \leq S, \end{cases} \tag{2}$$

where the equality $=_d$ is in distribution and the random variables W and D on the right hand side of Eq. (2) are independent of each other.

In this paper, we focus on a performance measure that is critical to the analysis of the (s, S) system - the *average undershoot below the reorder point s*. Define $\Delta = S - s$. The average undershoot can be used to estimate performance measures such as the average order size and the average number of periods between two successive orders:

$$\text{Average order size} = \Delta + \text{Average undershoot below } s,$$

$$\text{Average number of periods between two successive orders} = \frac{\Delta + \text{Average undershoot below } s}{\text{Average demand in one period}}.$$

The classical approach to evaluating the average undershoot of W is through renewal theory. The reader is referred to the books of Asmussen [1] and Tijms [21] for a discussion on renewal theory. For the exponential demand distribution with mean μ_1, Karlin [11] showed that W is also exponentially distributed with the same mean. Hence,

$$\text{Average undershoot below } s \text{ for exponential demand distribution with mean } \mu_1 = \mu_1.$$

For general distributions, it is however difficult to estimate the undershoot distribution in closed form. Instead, one must resort to numerical methods, approximations or bounds. A detailed description of numerical methods for this problem can be found in Sahin [17] and Tortorella [20]. For example, the computation of the average undershoot for gamma distributed demands can be done using a truncation of an infinite series combined with numerical integration methods. A simple and popular approximation that is very useful is based on the knowledge of the first two moments of demand. Assume the average value of demand is μ_1 and the second moment of demand is μ_2. For non-arithmetic distributions, the two moment approximation is provided in Feller [7]:

$$\text{Average undershoot below } s \quad \approx \quad \frac{\mu_2}{2\mu_1} \quad \text{for } \Delta >> \mu_1. \tag{3}$$

For discrete distributions on nonnegative integers, a factor of $1/2$ must is subtracted in the asymptotic approximation (see Hill [9]). Asymptotically, the approximation becomes tight as the value of Δ increases. Tijms and Groenevelt [22] use this approximation to estimate a recorder point s such that a specified fraction of demand is met directly from stock on hand. The approximation is exact for exponentially distributed demand distributions for all values of Δ. Using an extensive set of numerical experiments, Baganha, Pyke and Ferrer [2] tested the quality of the approximation for different demand distributions. They found the errors in the approximation to be significant for small values of Δ, low coefficients of variation and discrete distributions. To account for the errors, Smeitnik and Dekker [18] extended the approximation by incorporating the distribution function of the demand near zero. As an alternative, moment based upper and lower bounds have also been developed on the average undershoot. Barlow and Proschan [3] and Lorden [14] proposed the respective lower and upper bounds:

$$(\mu_1 - \Delta)^+ \quad \leq \quad \text{Average undershoot below } s \quad \leq \quad \frac{\mu_2}{\mu_1} \quad \text{for all } \Delta > 0. \tag{4}$$

Asymptotically the bounds are weaker than (3) for non-arithmetic distributions. However they remain valid uniformly over all values of Δ.

2 Main Results

In this paper, we provide new analytical lower and upper bounds in the flavor of Eq. (4) on the average undershoot in the steady state (s, S) inventory system. The bounds use moments of order up to three. To construct the bounds, we use a proof technique from the seminal work of Kingman [13] who derived a closed form upper bound on the steady state average waiting time in a $GI/GI/1$ queue using mean and variance information of the service and interarrival times. His bound was developed by equating the first two moments of a recursive distributional equation governing the steady state waiting time. In a similar vein, we relax the equality of distributions in (2) to the equality of the first few moments. Bertsimas and Natarajan [4] extended Kingman's approach to $GI/GI/1$ and $GI/GI/c$ queues by using higher order moments of the service and interarrival times. Their approach is computational in nature and makes use of the strong connection between the theory of moments and semidefinite optimization (see Bertsimas and Popescu [5] and Lasserre [12]). Osogami and Raymond [16] have recently used this approach in the transient analysis of $GI/GI/1$ queues. In the inventory context, our results are related to the work of Hu, Nananukul and Gong [8] who considered the infinite system of linear equations obtained from equating moments of all orders in Eq. (2). They developed a computational algorithm to compute performance measures for the steady-state inventory system using finite size approximations to an infinite system of linear equations.

The main results in this paper are provided next. The proofs are provided in Section 3.

Proposition 1. *Consider a steady-state (s,S) inventory system. The nonnegative random demand D has a finite mean $\mu_1 = E[D] > 0$ and a finite second moment $\mu_2 = E[D^2]$ with $\mu_2 \geq \mu_1^2$. Then the following hold:*

(a) A lower bound on the average undershoot is given by:

$$\textit{Average undershoot} \geq (\mu_1 - \Delta)^+, \tag{5}$$

(b) An upper bound on the average undershoot is given by:

$$\textit{Average undershoot} \leq \frac{\mu_2 + \sqrt{(\mu_2 + 2\mu_1\Delta)^2 - 8\mu_1^3\Delta}}{2\mu_1}. \tag{6}$$

(c) *These bounds are sharp. For $\mu_2 = \mu_1^2$ and $\Delta \leq \mu_1/2$, the lower and upper bounds are equal to $\mu_1 - \Delta$ and attained by the one point distribution:*

$$D = \left\{ \mu_1, \quad \textit{with probability } 1. \right.$$

With $\mu_2 = \mu_1^2$ and $\Delta = \mu_1/2$, the improvement in this new bound is:

$$\frac{\textit{Sharp two moment upper bound in Eq. } (6)}{\textit{Lorden's two moment upper bound}} = \frac{1}{2}.$$

Discussion: While the lower bound in Proposition 1 coincides with the lower bound in Eq. (4), the upper bound is new. The asymptotic performance of the new upper bound is clearly weak as it goes to ∞ with $\Delta \to \infty$. This in contrast to the asymptotic approximation and Lorden's upper bound which are bounded for all finite values of Δ. However for small values of Δ, the new upper bound can be sharper than Lorden's upper bound. It is important to note that in this regime the asymptotic approximation also performs poorly (refer to Baganha, Pyke and Ferrer [2]). Another feature of Lorden's upper bound is that it is independent of Δ. Lorden's bound is sharp when demand is a one point distribution with $P(D = \mu_1) = 1$ and $\Delta = k\mu_1$ for any nonnegative integer k (see Chang [6]). On the other hand, the new upper bound depends on the value of Δ and is sharp when demand is a one point distribution with $P(D = \mu_1) = 1$ and Δ takes values below $\mu_1/2$. A straightforward comparison of the two bounds results in the following corollary which states that the new upper bound is tighter for small values of Δ and low values of coefficient of variation. .

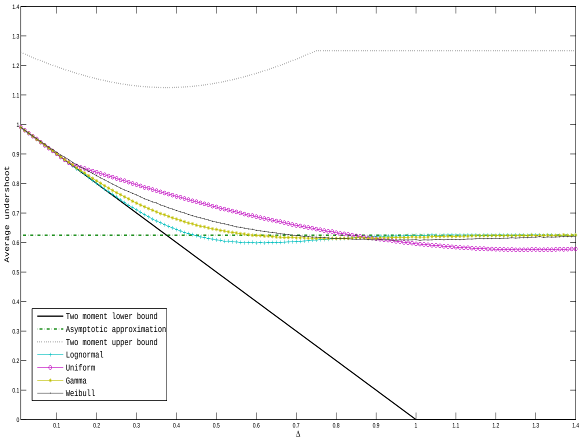
Corollary 1. *Define the coefficient of variation as $cv = \sigma/\mu_1$ where σ is the standard deviation of demand. The two moment upper bound on the average undershoot in (6) is strictly smaller than Lorden's upper bound in Eq. (4) iff*

$$0 \quad < \quad \Delta \quad < \quad \mu_1(1 - cv^2). \tag{7}$$

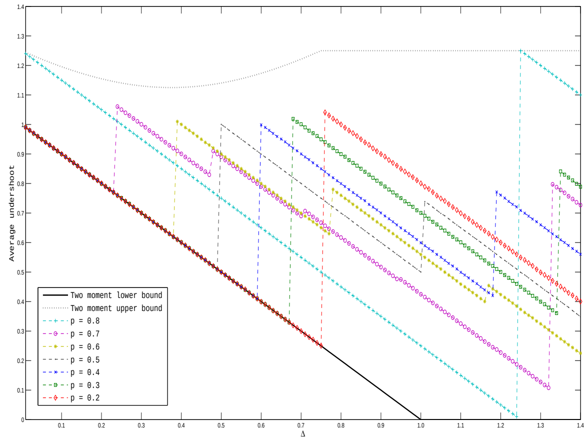
Numerical Example: Consider five classes of nonnegative demand distributions with mean $\mu_1 = 1$ and coefficient of variation $cv = 0.5$: (a) Lognormal (b) Uniform (c) Gamma, (d) Weibull and (e) Two point distributions. The two point distribution is defined as:

$$D = \begin{cases} \mu_1 \left(1 - cv\sqrt{\dfrac{p}{1-p}}\right), & \text{with probability } 1 - p, \\ \mu_1 \left(1 + cv\sqrt{\dfrac{1-p}{p}}\right), & \text{with probability } p, \end{cases}$$

where $p \leq 1/(1 + cv^2)$ ensures nonnegativity of demand. Figure 1(a) and 1(b) plots the average undershoot where the two moment upper bound is the minimum of the new bound and Lorden's upper bound. Compared to continuous distributions, the discrete distributions exhibit greater oscillatory behavior as documented in [2]. The U shaped portion of the upper bound corresponds to the case when the new upper bound is tighter. The improvement in the bound is sharper for discrete distributions.



(a) Continuous distributions



(b) Two point distributions

Figure 1: Average undershoot with $\mu_1 = 1$, $cv = 0.5$. Asymptotic approximation (continuous) = 0.625, Lorden's upper bound = 1.25

The next proposition provides new three moment bounds by incorporating the skewness information on the demand.

Proposition 2. *Consider a steady-state (s,S) inventory system. The nonnegative random demand D has a finite mean $\mu_1 = E[D] > 0$, a finite second moment $\mu_2 = E[D^2] \geq \mu_1^2$ and a finite third moment $\mu_3 = E[D^3] \geq \mu_2^2/\mu_1$. Then the following hold:*

(a) A lower bound on the average undershoot is given by:

$$\text{Average undershoot} \geq \begin{cases} \dfrac{\Delta(\mu_2 - \Delta\mu_1)^2}{\mu_1\mu_3 - \mu_2^2 + \Delta\mu_1(\mu_2 - \Delta\mu_1)} & \text{if } \Delta' < \Delta < \mu_2/\mu_1 \\ (\mu_1 - \Delta)^+ & \text{otherwise,} \end{cases} \tag{8}$$

where

$$\Delta' = \frac{\mu_3 - \mu_2\mu_1 - \sqrt{(\mu_3 - \mu_2\mu_1)^2 - 4(\mu_2 - \mu_1^2)(\mu_1\mu_3 - \mu_2^2)}}{2(\mu_2 - \mu_1^2)}. \tag{9}$$

(b) An upper bound on the average undershoot is given by:

$$\text{Avg undershoot} \leq \min\left(\frac{\mu_2 + \sqrt{(\mu_2 + 2\mu_1\Delta)^2 - 8\mu_1^3\Delta}}{2\mu_1}, \frac{3\mu_2 - \mu_1\Delta + \sqrt{(\mu_2 - 3\mu_1\Delta)^2 + 16(\mu_1\mu_3 - \mu_2^2)}}{4\mu_1}\right). \tag{10}$$

(c) These bounds are sharp. For $\mu_3 = \mu_2^2/\mu_1$ and $\Delta \leq \mu_2/3\mu_1$, the lower and upper bounds are equal to $\mu_2/\mu_1 - \Delta$ and attained by the two point distribution:

$$D = \begin{cases} 0, & \text{with probability } 1 - \dfrac{\mu_1^2}{\mu_2}, \\ \dfrac{\mu_2}{\mu_1}, & \text{with probability } \dfrac{\mu_1^2}{\mu_2}. \end{cases}$$

With $\mu_3 = \mu_2^2/\mu_1$*,* $\mu_2 = 3\mu_1^2/2$ *and* $\Delta = \mu_1/2$*,, the improvement in this new bound is:*

$$\frac{\textit{Sharp three moment upper bound in Eq. } (10)}{\textit{Two moment upper bound in Eq. } (6)} = \frac{2}{3}.$$

Discussion: Unlike the two moment bound, we do not know of any closed form three moment bound in the inventory context. As μ_3 decreases (or the demand gets more negatively skewed), the three moment upper bound decreases. Similarly as the demand gets more negatively skewed, the three moment lower bound increases. The sharp instance in case (c) is when the demand is most negatively skewed with the given first and second moments.

Numerical Example: Table 1 provides a comparison of two moment and three moment bounds by varying Δ and the coefficient of skewness which is defined as $cs = (2\mu_1^3 - 3\mu_1\mu_2 + \mu_3)/(\mu_2 - \mu_1^2)^{3/2}$. The three moment bounds provide the most improvement for negatively skewed demands. For $cs = -1.5$, the three moment bounds are tight for values of Δ below 0.4167. This corresponds to the two point distribution:

$$D = \begin{cases} 0, & \text{with probability } 0.2, \\ 1.25, & \text{with probability } 0.8. \end{cases}$$

Table 1: Average undershoot bounds with $\mu_1 = 1$, $cv = 0.5$. Asymptotic approximation (continuous): 0.625, Lorden upper bound: 1.25

cs	Bound	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
-1.50	Three moment lower	1.1500	1.0500	0.9500	0.8500	0.7500	0.6500	0.5500	0.4500
	Three moment upper	1.1500	1.0500	0.9500	0.8500	0.8750	0.9250	0.9750	1.0250
0.00	Three moment lower	0.9000	0.8000	0.7000	0.6000	0.5000	0.4390	0.3699	0.2959
	Three moment upper	1.1956	1.1547	1.1306	1.1256	1.1404	1.1733	1.2213	1.2573
0.25	Three moment lower	0.9000	0.8000	0.7000	0.6000	0.5000	0.4164	0.3507	0.2799
	Three moment upper	1.1956	1.1547	1.1306	1.1256	1.1404	1.1733	1.2213	1.2812
	Two moment lower	0.9000	0.8000	0.7000	0.6000	0.5000	0.4000	0.3000	0.2000
	Two moment upper	1.1956	1.1547	1.1306	1.1256	1.1404	1.1733	1.2213	1.2812

3 Proofs

To simplify the derivations, define $Z = S - W$. Eq. (2) can then be rewritten as:

$$Z =_d \begin{cases} D & \text{if } Z > \Delta \\ Z + D & \text{if } 0 \leq Z \leq \Delta. \end{cases} \tag{11}$$

We define the support sets as $\mathcal{O} = \{Z \mid Z > \Delta\}$ and $\overline{\mathcal{O}} = \{Z \mid 0 \leq Z \leq \Delta\}$. Let η_Δ denote the undershoot of the steady state inventory level below the reorder point s. The average undershoot is evaluated as:

$$E[\eta_\Delta] \quad = \quad E\left[Z\middle|\mathcal{O}\right] - \Delta. \tag{12}$$

Proof of Proposition 1. Define the two sets of decision variables as the scaled conditional moments:

$$x_i = E\left[Z^i|\mathcal{O}]P[\mathcal{O}\right] \quad \text{and} \quad y_i = E\left[Z^i|\overline{\mathcal{O}}]P[\overline{\mathcal{O}}\right] \quad \text{for} \quad i = 0, 1, 2.$$

The scaled conditional moments satisfy the following set of conditions:

$$x_0 + y_0 = 1 \tag{13}$$
$$x_1 = \mu_1 \tag{14}$$
$$x_2 = \mu_2 + 2\mu_1 y_1 \tag{15}$$
$$\begin{pmatrix} x_0 & x_1 \\ x_1 & x_2 \end{pmatrix} \succeq 0 \tag{16}$$
$$x_1 \geq \Delta x_0 \tag{17}$$
$$\begin{pmatrix} y_0 & y_1 \\ y_1 & y_2 \end{pmatrix} \succeq 0 \tag{18}$$
$$y_2 \leq \Delta y_1. \tag{19}$$

Eq. (13) is obtained by observing that the sets $\mathcal{O}$ and $\overline{\mathcal{O}}$ form a partition of $\Re^+$. Eqs. (14)-(15) are derived by equating the first two moments of the random variables on the left and right hand side of Eq. (11). Eqs. (16)-(17) are the two moment feasibility conditions for a univariate random variable over the semi-infinite interval $\{Z \mid Z \geq \Delta\}$ where we extend the definition of the set $\mathcal{O}$ to include $\{Z = \Delta\}$. These conditions would still form necessary moment conditions. This is the standard semidefinite representation for the feasibility of the Stieltjes moment problem (see Nesterov [15]). Eqs. (18)-(19) are the standard Hausdorff moment feasibility conditions for the finite interval $\overline{\mathcal{O}}$.
It is easy to verify that $x_0 \neq 0$. Else, the positive semidefiniteness condition in (16) would force $x_1 = 0$. This would make Eq. (14) infeasible given that $\mu_1 > 0$. Hence, from Eq. (12), the average undershoot can be defined in terms of the variable x_0 as:

$$E[\eta_\Delta] \quad = \quad \frac{\mu_1}{x_0} - \Delta. \tag{20}$$

To obtain lower and upper bounds on the average undershoot, we obtain upper and lower bounds on x_0 satisfying constraints (13)-(19). These problems can be formulated as semidefinite optimization problems over the six decision variables (x_0, x_1, x_2) and (y_0, y_1, y_2):

$$\begin{array}{lll} (\mathbf{P}_2) & \max\ (\min) & x_0 \\ & \text{s.t.} & \text{Eq. (13)} - \text{Eq. (19)}. \end{array}$$

We solve the semidefinite programs in closed form next.
Lower bound on $E[\eta_\Delta]$: Using constraints (14) and (17) in conjunction with definition of x_0 as a

probability implies that $x_0 \leq \min(1, \mu_1/\Delta)$. This forms a valid upper bound on x_0 for all feasible solutions satisfying (13)-(19). We show that this upper bound is tight for the maximization version of the semidefinite program ($\mathbf{P}_2$) by identifying a feasible solution that attains the bound. The feasible solution is defined as follows:

(i) If $\mu_1 \geq \Delta$, set:

$$(x_0, x_1, x_2) = (1, \mu_1, \mu_2) \quad \text{and} \quad (y_0, y_1, y_2) = (0, 0, 0)\,.$$

It is easy to check that this solution is feasible for the set of constraints (13)-(19) and attains the upper bound of 1.

(ii) If $\mu_1 \leq \Delta$, set:

$$(x_0, x_1, x_2) = \left(\frac{\mu_1}{\Delta}, \mu_1, \mu_2 + 2\mu_1(\Delta - \mu_1)\right) \quad \text{and} \quad (y_0, y_1, y_2) = \left(\frac{\Delta - \mu_1}{\Delta}, \Delta - \mu_1, (\Delta - \mu_1)\Delta\right).$$

We focus on the positive semidefinite constraint in Eq. (16) since the feasibility of all other constraints is easily verifiable. Note that:

$$\begin{aligned} x_0 x_2 - x_1^2 &= \frac{\mu_1}{\Delta}\left(\mu_2 + 2\mu_1(\Delta - \mu_1)\right) - \mu_1^2 \\ &= \frac{\mu_1}{\Delta}\left((\mu_2 - \mu_1^2) + \mu_1(\Delta - \mu_1)\right) \\ &\geq 0 \qquad\qquad (\text{Since } \mu_2 \geq \mu_1^2 \text{ and } \Delta \geq \mu_1 > 0). \end{aligned}$$

The nonnegativity $x_0x_2 - x_1^2$ along with the nonnegativity of x_0 implies the positive semidefiniteness of the matrix in Eq. (16). This feasible solution attains the upper bound of μ_1/Δ.

This proves the tightness of the two moment lower bound on the expected undershoot $\eta^{(L)}_{\Delta,2} = (\mu_1 - \Delta)^+$ based on ($\mathbf{P}_2$).

Upper bound on $E[\eta_\Delta]$: To evaluate a two moment lower bound on x_0, we use the following sequence of relationships:

$$\begin{aligned} x_0 &\geq \frac{x_1^2}{x_2} && (\text{From Eq. (16) and since } x_2 > 0) \\ &= \frac{\mu_1^2}{\mu_2 + 2\mu_1 y_1} && (\text{From Eq. (14) and (15)}) \\ &\geq \frac{\mu_1^2}{\mu_2 + 2\mu_1\Delta(1 - x_0)} && (\text{Since } 0 \leq y_1 \leq \Delta y_0 \text{ from Eq. (18) and (19) and } y_0 = 1 - x_0). \end{aligned}$$

Hence, the variable x_0 satisfies the quadratic inequality:

$$q(x_0) := -2\mu_1\Delta x_0^2 + (\mu_2 + 2\mu_1\Delta)x_0 - \mu_1^2 \geq 0.$$

For $x_0 = 0$, the inequality is violated since $q(0) = -\mu_1^2 < 0$. For $x_0 = \min(1, \mu_1/\Delta)$, the inequality is satisfied since,

(i) If $\mu_1 \geq \Delta$, the value $q(1) = \mu_2 - \mu_1^2$ is strictly positive,

(ii) If $\mu_1 \leq \Delta$, the value $q(\mu_1/\Delta) = \mu_1\left((\mu_2 - \mu_1^2) + \mu_1(\Delta - \mu_1)\right)/\Delta$ is strictly positive. The probability of ordering must be greater that the smaller root of the quadratic equation $q(\cdot) = 0$. This provides a lower bound on x_0:

$$\begin{aligned} x_0 &\geq \frac{\mu_2 + 2\mu_1\Delta - \sqrt{(\mu_2 + 2\mu_1\Delta)^2 - 8\mu_1^3\Delta}}{4\mu_1\Delta} \\ &= \frac{2\mu_1^2}{\mu_2 + 2\mu_1\Delta + \sqrt{(\mu_2 + 2\mu_1\Delta)^2 - 8\mu_1^3\Delta}}. \end{aligned}$$

Denoting this lower bound as $\underline{x}_0$, tightness for the semidefinite program is shown by generating a feasible solution to $(\mathbf{P}_2)$ that attains the bound. The solution is:

$$(x_0, x_1, x_2) = (\underline{x}_0, \mu_1, \mu_2 + 2\mu_1\Delta(1 - \underline{x}_0)) \quad \text{and} \quad (y_0, y_1, y_2) = \left(1 - \underline{x}_0, \Delta(1 - \underline{x}_0), \Delta^2(1 - \underline{x}_0)\right).$$

It can be verified that this satisfies all the constraints. This proves the tightness of the two moment upper bound on the expected undershoot $\eta_{\Delta,2}^{(U)} = (\mu_2 + \sqrt{(\mu_2 + 2\mu_1\Delta)^2 - 8\mu_1^3\Delta})/2\mu_1$ based on the proposed semidefinite program. ■

The next proof tightens the two moment bound by incorporating the third moment information of the demand. The proof of this result makes use of semidefinite programming duality.

Proof of Proposition 2(a). We define the sets of decision variables as the scaled conditional moments:

$$x_i = E\left[Z^i|\mathcal{O}]P[\mathcal{O}\right] \quad \text{and} \quad y_i = E\left[Z^i|\overline{\mathcal{O}}]P[\overline{\mathcal{O}}\right] \quad \text{for} \quad i = 0, 1, 2, 3.$$

An upper bound on the probability of ordering in an infinite horizon (s, S) inventory system is the optimal objective value of the semidefinite program:

$$\begin{aligned} (\mathbf{P}_3) \quad \max \quad & x_0 \\ \text{s.t.} \quad & \text{Eq. (13)} - \text{Eq. (15)} \\ & x_3 = \mu_3 + 3\mu_2 y_1 + 3\mu_1 y_2 && (21) \\ & \begin{pmatrix} x_1 & x_2 \\ x_2 & x_3 \end{pmatrix} \succeq \Delta \begin{pmatrix} x_0 & x_1 \\ x_1 & x_2 \end{pmatrix} \succeq 0 && (22) \\ & \Delta \begin{pmatrix} y_0 & y_1 \\ y_1 & y_2 \end{pmatrix} \succeq \begin{pmatrix} y_1 & y_2 \\ y_2 & y_3 \end{pmatrix} \succeq 0 && (23) \end{aligned}$$

Constraints (13)-(15) are from the two moment formulation. Constraint (21) is obtained by equating the third moments of the random variables in Eq. (11). Constraints (22)-(23) are the three moment feasibility conditions for a univariate random variable over a semi-infinite and finite interval (see Nesterov [15]). We start by identifying the region for Δ in which the two moment upper bound derived in Proposition 1 $x_0 \leq \min(1, \mu_1/\Delta)$, is still the optimal objective value for the semidefinite $(\mathbf{P}_3)$. This corresponds to cases (i) and (ii) discussed next.

(i) Consider the upper bound of $x_0 = 1$. The unique values of the decision variables that result in this bound and satisfy the constraints in ($\mathbf{P}_3$) are:

$$(x_0, x_1, x_2, x_3) = (1, \mu_1, \mu_2, \mu_3) \quad \text{and} \quad (y_0, y_1, y_2, y_3) = (0, 0, 0, 0)\,.$$

To ensure that the positive semidefiniteness constraints in Eq. (22) are satisfied, the following inequalities must hold:

$$\mu_1 \geq \Delta \quad \text{and} \quad (\mu_1 - \Delta)(\mu_3 - \Delta\mu_2) \geq (\mu_2 - \Delta\mu_1)^2.$$

The first condition is satisfied for all $\Delta \in [0, \mu_1]$ while the second condition is satisfied for $\Delta = 0$ but violated for $\Delta = \mu_1$. Hence, the upper bound on x_0 is valid for all values of Δ that is lesser than or equal to the smaller root of the quadratic equation:

$$q'(\Delta) := (\mu_2 - \mu_1^2)\Delta^2 + (\mu_2\mu_1 - \mu_3)\Delta + (\mu_3\mu_1 - \mu_2^2) = 0.$$

The valid range for Δ is $\Delta \leq \Delta'$ where Δ' is the smaller root of the quadratic equation $q'(\cdot) = 0$ and defined in Eq. (9).

(ii) Consider the upper bound $x_0 = \mu_1/\Delta$. Since $x_1 = \mu_1 = \Delta x_0$, the positive semidefiniteness condition in Eq. (22) implies that:

$$x_2 = \Delta x_1 = \Delta\mu_1.$$

The equality of the second moments in Eq. (15) implies:

$$y_1 = \frac{\Delta\mu_1 - \mu_2}{2\mu_1}.$$

Since the y_1 variable is nonnegative, Δ must satisfy $\Delta \geq \frac{\mu_2}{\mu_1}$. A feasible solution to ($\mathbf{P}_3$) that attains this bound is:

$$\begin{aligned}
(x_0, x_1, x_2, x_3) &= \left(\frac{\mu_1}{\Delta}, \mu_1, \Delta\mu_1, \mu_3 + 3\left(\frac{\Delta^2\mu_1^2 - \mu_2^2}{2\mu_1}\right)\right), \\
(y_0, y_1, y_2, y_3) &= \left(\frac{\Delta - \mu_1}{\Delta}, \frac{\Delta\mu_1 - \mu_2}{2\mu_1}, \Delta\left(\frac{\Delta\mu_1 - \mu_2}{2\mu_1}\right), \Delta^2\left(\frac{\Delta\mu_1 - \mu_2}{2\mu_1}\right)\right).
\end{aligned}$$

Since $x_1 = \Delta x_0$ and $x_2 = \Delta x_1$, the first positive semidefinite constraint in Eq. (22) is verifiable as:

$$\begin{aligned}
x_3 - \Delta x_2 &= \frac{2\mu_1\mu_3 + \Delta^2\mu_1^2 - 3\mu_2^2}{2\mu_1} \\
&= \frac{2\left(\mu_1\mu_3 - \mu_2^2\right) + \left(\Delta^2\mu_1^2 - \mu_2^2\right)}{2\mu_1} \\
&\geq 0 \qquad\qquad (\text{Since } \mu_1\mu_3 \geq \mu_2^2 \text{ and } \Delta \geq \mu_2/\mu_1).
\end{aligned}$$

All other constraints can be easily verified.

(iii) We analyze the semidefinite program for $\Delta' < \Delta < \mu_2/\mu_1$ and derive a new bound in this interval. We show that an optimal solution is given by:

$$\begin{aligned}(x_0,x_1,x_2,x_3) &= \left(\frac{\mu_1}{\Delta} - \frac{(\mu_2-\Delta\mu_1)^2}{\Delta(\mu_3-\Delta\mu_2)}, \mu_1,\mu_2,\mu_3\right),\\ (y_0,y_1,y_2,y_3) &= \left(1-\frac{\mu_1}{\Delta}+\frac{(\mu_2-\Delta\mu_1)^2}{\Delta(\mu_3-\Delta\mu_2)}, 0,0,0\right).\end{aligned} \tag{24}$$

To prove it, we start by formulating the dual $(\mathbf{D}_3)$ of the semidefinite program $(\mathbf{P}_3)$:

$$(\mathbf{D}_3)\quad \min \quad z_1+\mu_1 z_2+\mu_2 z_3+\mu_3 z_4$$

$$\text{s.t.}\quad \begin{pmatrix} z_1+\Delta z_5-1 & \frac{1}{2}z_2-\frac{1}{2}z_5+\frac{\Delta}{2}z_6\\ \frac{1}{2}z_2-\frac{1}{2}z_5+\frac{\Delta}{2}z_6 & z_3+\Delta z_4 - z_6\end{pmatrix}\succeq 0 \tag{25}$$

$$\begin{pmatrix} z_5 & \frac{1}{2}z_6\\ \frac{1}{2}z_6 & z_4\end{pmatrix}\succeq 0 \tag{26}$$

$$\begin{pmatrix} \frac{1}{\Delta}z_1-2\mu_1 z_3-3\mu_2 z_4 - z_8 & -\frac{3}{2}\mu_1 z_4-\frac{1}{2}z_7+\frac{1}{2\Delta}z_8\\ -\frac{3}{2}\mu_1 z_4-\frac{1}{2}z_7+\frac{1}{2\Delta}z_8 & \frac{1}{\Delta}z_7\end{pmatrix}\succeq 0 \tag{27}$$

$$\begin{pmatrix} z_1 & \frac{1}{2}z_8\\ \frac{1}{2}z_8 & z_7\end{pmatrix}\succeq 0 \tag{28}$$

The proof consists of (a) checking primal feasibility and (b) using complementary slackness to identify a dual solution and checking for dual feasibility. From weak duality for semidefinite programming this would imply the optimality of the primal and dual solutions.

Step (a): Primal feasibility

To validate the feasibility of the solution (24) for the semidefinite program $(\mathbf{P}_3)$, we start by verifying that x_0 is strictly positive:

$$\begin{aligned} x_0 &= \frac{\left(\mu_1\mu_3-\mu_2^2\right)+\Delta\mu_1\left(\mu_2-\Delta\mu_1\right)}{\Delta(\mu_3-\Delta\mu_2)}\\ &> 0 \qquad\qquad (\text{Since } \mu_1\mu_3\geq\mu_2^2 \text{ and } \Delta<\mu_2/\mu_1\leq\mu_3/\mu_2).\end{aligned}$$

Checking for $x_0 < 1$ is equivalent to verifying the quadratic inequality $q'(\Delta) < 0$ which is guaranteed in the range $\Delta \in (\Delta', \mu_2/\mu_1)$. The second positive semidefinite constraint in Eq. (22) is verified as:

$$x_0x_2 - x_1^2 = \frac{\left(\mu_1\mu_3-\mu_2^2\right)\left(\mu_2-\Delta\mu_1\right)}{\Delta(\mu_3-\Delta\mu_2)} \geq 0.$$

To verify the first positive semidefinite constraint in Eq. (23), note that:

$$(x_1-\Delta x_0) = \frac{\left(\mu_2-\Delta\mu_1\right)^2}{(\mu_3-\Delta\mu_2)} > 0.$$

Also $x_3-\Delta x_2 = \mu_3-\Delta\mu_2 > 0$ in this range. Finally,

$$(x_1-\Delta x_0)(x_3-\Delta x_2)-(x_2-\Delta x_1)^2 = 0.$$

The feasibility of all other constraints in the primal semidefinite program is easy to verify.

Step (b): Complementary slackness and dual feasibility

The complementary slackness condition for $(\mathbf{P}_3)$ and $(\mathbf{D}_3)$ given the primal feasible solution is:

$$\begin{pmatrix} \frac{\mu_1}{\Delta} - \frac{(\mu_2-\Delta\mu_1)^2}{\Delta(\mu_3-\Delta\mu_2)} & \mu_1 \\ \mu_1 & \mu_2 \end{pmatrix} \begin{pmatrix} z_1 + \Delta z_5 - 1 & \frac{1}{2}z_2 - \frac{1}{2}z_5 + \frac{\Delta}{2}z_6 \\ \frac{1}{2}z_2 - \frac{1}{2}z_5 + \frac{\Delta}{2}z_6 & z_3 + \Delta z_4 - z_6 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \tag{29}$$

$$\begin{pmatrix} \frac{(\mu_2-\Delta\mu_1)^2}{\mu_3-\Delta\mu_2} & \mu_2 - \Delta\mu_1 \\ \mu_2 - \Delta\mu_1 & \mu_3 - \Delta\mu_2 \end{pmatrix} \begin{pmatrix} z_5 & \frac{1}{2}z_6 \\ \frac{1}{2}z_6 & z_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \tag{30}$$

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\Delta}z_1 - 2\mu_1 z_3 - 3\mu_2 z_4 - z_8 & -\frac{3}{2}\mu_1 z_4 - \frac{1}{2}z_7 + \frac{1}{2\Delta}z_8 \\ -\frac{3}{2}\mu_1 z_4 - \frac{1}{2}z_7 + \frac{1}{2\Delta}z_8 & \frac{1}{\Delta}z_7 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \tag{31}$$

$$\begin{pmatrix} 1 - \frac{\mu_1}{\Delta} + \frac{(\mu_2-\Delta\mu_1)^2}{\Delta(\mu_3-\Delta\mu_2)} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} z_1 & \frac{1}{2\Delta}z_8 \\ \frac{1}{2\Delta}z_8 & z_7 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \tag{32}$$

Since $y_0 > 0$ for $\Delta' < \Delta < \mu_2/\mu_1$, condition (32) gives us that $z_1 = 0$. The positive semidefiniteness condition (28) implies $z_8 = 0$. The positive definiteness of the matrix

$$\begin{pmatrix} \frac{\mu_1}{\Delta} - \frac{(\mu_2-\Delta\mu_1)^2}{\Delta(\mu_3-\Delta\mu_2)} & \mu_1 \\ \mu_1 & \mu_2 \end{pmatrix} \succ 0,$$

implies that from condition (25) and (29)

$$\begin{pmatrix} \Delta z_5 - 1 & \frac{1}{2}z_2 - \frac{1}{2}z_5 + \frac{\Delta}{2}z_6 \\ \frac{1}{2}z_2 - \frac{1}{2}z_5 + \frac{\Delta}{2}z_6 & z_3 + \Delta z_4 - z_6 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{33}$$

Hence $z_5 = 1/\Delta$. Plugging this into (30) allows us to solve for z_6 and z_4:

$$z_6 = \frac{2(\Delta\mu_1 - \mu_2)}{\Delta(\mu_3 - \Delta\mu_2)} \quad \text{and} \quad z_4 = \frac{(\Delta\mu_1 - \mu_2)^2}{\Delta(\mu_3 - \Delta\mu_2)^2}.$$

Substituting back into (33), we get

$$z_2 = \frac{1}{\Delta} + \frac{2(\mu_2 - \Delta\mu_1)}{\mu_3 - \Delta\mu_2} \quad \text{and} \quad z_3 = \frac{2(\Delta\mu_1 - \mu_2)}{\Delta(\mu_3 - \Delta\mu_2)} - \frac{(\Delta\mu_1 - \mu_2)^2}{(\mu_3 - \Delta\mu_2)^2}.$$

Under these choices of dual variables, the positive semidefinite constraints (25) and (26) are satisfied. The final step in the derivation of the optimal dual solution is to choose a z_7 such that the constraints (27) and (28) are satisfied. This is equivalent to $z_7 \geq 0$ and the determinant of the matrix in (27) is nonnegative. The latter condition is equivalent to the following quadratic function being nonnegative

$$q''(z_7) := az_7^2 + bz_7 + c \geq 0, \tag{34}$$

where

$$\begin{aligned} a &= -1 \\ b &= \frac{(\Delta\mu_1 - \mu_2)}{\Delta^2(\mu_3 - \Delta\mu_2)^2}\left(2\Delta\mu_1\mu_2 - 16\mu_1\mu_3 + 2\Delta^2\mu_1^2 + 12\mu_2^2\right) \\ c &= -9\mu_1^2\frac{(\Delta\mu_1 - \mu_2)^4}{\Delta^2(\mu_3 - \Delta\mu_2)^4}. \end{aligned}$$

At $z_7 = 0$, $q''(0) = c < 0$ while the value of the derivative is positive

$$\begin{aligned}
\left.\frac{dq''(z)}{dz}\right|_{z=0} &= b \\
&= 2\mu_1 \frac{(\Delta\mu_1 - \mu_2)}{\Delta^2(\mu_3 - \Delta\mu_2)^2}\big((\Delta\mu_2 - \mu_3) + (\Delta^2\mu_1 - \mu_3)\big) \\
&> 0 \qquad \text{(Since } \mu_3 > \Delta\mu_2 > \Delta^2\mu_1\text{)}.
\end{aligned}$$

Lastly the discriminant of the quadratic is nonnegative

$$b^2 - 4ac \;=\; 16\mu_1^2 \frac{(\Delta\mu_1 - \mu_2)^2}{\Delta^4(\mu_3 - \Delta\mu_2)^4}(\Delta\mu_2 - \mu_3)(\Delta^2\mu_1 - \mu_3) \;>\; 0, \tag{35}$$

implying that there exist real roots. Hence there exists a nonnegative z_7 that satisfies $q''(z_7) \geq 0$. Thus, (27) and (28) are satisfied. Since the pair of primal and dual feasible solutions satisfy the complementary slackness conditions, it is optimal.

Putting together the three cases, provides the lower bound on the average undershoot in Eq. (8). ■

Proof of Proposition 2(b) In the three moment case, a lower bound on the probability of ordering in an infinite horizon (s, S) inventory system is the optimal objective value of the following semidefinite program:

$$\begin{aligned}
(\mathbf{P}_3) \quad \min \quad & x_0 \\
\text{s.t.} \quad & \text{Eq. (13) -- Eq. (15)}, \\
& \text{Eq. (21) -- Eq. (23)}.
\end{aligned}$$

To solve $(\mathbf{P}_3)$, we make use of the inequality:

$$x_0 \;\geq\; \frac{\mu_1^2}{x_2} \quad \text{(From Eq. (14), (22) and since } x_2 > 0\text{)} \tag{36}$$

Since our goal is to find a lower bound on x_0 or alternatively find an upper bound on x_2 , we derive the following valid inequality:

$$y_2 \leq \Delta y_1. \tag{37}$$

This inequality is obtained from the positive semidefiniteness matrices in Eq. (23):

$$y_2^2 \leq y_3 y_1 \leq \Delta y_2 y_1,$$

and observing that $y_2 = 0$ is equivalent to $y_1 = 0$ from these conditions. The upper bound on x_2 is then derived as follows:

$$\begin{aligned}
x_2^2 &\leq \mu_1 x_3 && \text{(From Eq. (14), and positive semidefiniteness in (22))} \\
&= \mu_1(\mu_3 + 3\mu_2 y_1 + 3\mu_1 y_2) && \text{(From Eq. (21))} \\
&\leq \mu_1\mu_3 + (3\mu_2 + 3\mu_1\Delta)\,\mu_1 y_1 && \text{(From Eq. (37))} \\
&= \mu_1\mu_3 + (3\mu_2 + 3\mu_1\Delta)\left(\frac{x_2 - \mu_2}{2}\right) && \text{(From Eq. (15))}
\end{aligned}$$

For this inequality to hold, the value x_2 can be at most the larger (nonnegative) root of the quadratic equation:

$$q'''(x_2) := -2x_2^2 + (3\mu_2 + 3\mu_1\Delta)x_2 - (3\mu_2 + 3\mu_1\Delta)\mu_2 + 2\mu_1\mu_3 \quad = \quad 0.$$

The upper bound is given as:

$$x_2 \quad \leq \quad \frac{3\mu_2 + 3\mu_1\Delta + \sqrt{(3\mu_2 + 3\mu_1\Delta)^2 - 8((3\mu_2 + 3\mu_1\Delta)\mu_2 - 2\mu_1\mu_3)}}{4}.$$

Substituting into Eq. (36), we obtain the lower bound on the probability of ordering,

$$x_0 \geq \frac{4\mu_1^2}{3\mu_2 + 3\mu_1\Delta + \sqrt{(3\mu_2 + 3\mu_1\Delta)^2 - 8[(3\mu_2 + 3\mu_1\Delta)\mu_2 - 2\mu_1\mu_3]}}. \tag{38}$$

The corresponding upper bound on the average undershoot is:

$$E[\eta_\Delta] \leq \frac{3\mu_2 - \mu_1\Delta + \sqrt{(3\mu_2 + 3\mu_1\Delta)^2 - 8((3\mu_2 + 3\mu_1\Delta)\mu_2 - 2\mu_1\mu_3)}}{4\mu_1}.$$

Combining with the two moment bound in Proposition 1 leads to Eq. (10). ■

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