

Constructing Risk Measures from Uncertainty Sets

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We illustrate the correspondence between uncertainty sets in robust optimization and some popular risk measures in finance and show how robust optimization can be used to generalize the concepts of these risk measures. We also show that by using properly defined uncertainty sets in robust optimization models, one can construct coherent risk measures and address the issue of the computational tractability of the resulting formulations. Our results have implications for efficient portfolio optimization under different measures of risk.

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1. Introduction

Markowitz (1952) was the first to model the important trade-off between risk and return in portfolio selection as an optimization problem. He suggested choosing an asset mix such that the portfolio variance is minimum for a fixed target level of expected return. It is now known (Tobin 1958, Chamberlain 1983) that the mean-variance framework is appropriate if the distribution of returns is elliptically symmetric (e.g., multivariate normal). In this case, the optimal mean-variance portfolio allocation is consistent with any set of preferences for market agents in the sense that given a fixed expected return, any investor will prefer the portfolio with the smallest standard deviation. However, when returns are not symmetrically distributed, or when a downside risk is more weighted than an upside risk, variance is not an accurate measure of investor risk preferences. Markowitz (1991) acknowledges this shortcoming and discusses alternative risk measures in a more general mean-risk approach. Such considerations and the theory of stochastic dominance (Levy 1992) spurred interest in asymmetric or quantile-based portfolio risk measures such as expectation of loss, semivariance, Value-at-Risk (VaR), and others (Jorion 2000, Dowd 1998, Konno and Yamazaki 1991, Carino and Turner 1998). Generalizations of these approaches to worst-case risk measures when the distributions and the parameters are themselves unknown have been studied for the variance and the VaR

risk measures (Halldorsson and Tutuncu 2003, Goldfarb and Iyengar 2003, El Ghaoui et al. 2003). Artzner et al. (1999) introduced an axiomatic methodology to characterize desirable properties in risk measures. They named risk measures that satisfied their four axioms *coherent*. A popular example of a coherent risk measure is Conditional Value-at-Risk (CVaR), discussed in Rockafellar and Uryasev (2000, 2002).

If one thinks of future asset returns as unknown parameters, one can view the portfolio problem as an optimization problem with uncertain coefficients. It is then natural to approach it with tools for optimization under uncertainty, such as the recently developed *robust optimization techniques*. The main idea in robust optimization is that the optimal solution must remain feasible for any realization of the uncertain parameters within a prespecified uncertainty set. The “size” and “shape” of the uncertainty sets are usually based on probability estimates on the quality of the solution. It has been observed (Ben-Tal and Nemirovski 1999) that by stating the portfolio optimization problem as one of maximizing return subject to the constraint that future returns varies in an ellipsoidal uncertainty set defined by the covariance structure of the uncertain returns, the robust counterpart of the portfolio optimization problem is reminiscent of the Markowitz formulation. This paper builds on this observation and presents a unified theory that relates portfolio risk measures to robust optimization uncertainty sets.

Most generally, our contribution is to bring together several important results in the theory of risk measures that have so far been disparate, and to add our perspective on computational tractability and definitions of risk measures in a new context—that of robust optimization. Our specific contributions can be summarized as follows:

(a) We show explicitly how risk measures such as standard deviation, worst-case VaR, and CVaR can be mapped to robust optimization uncertainty sets. Some of these results exist in the literature, but we present them in a unified framework, and further study how the idea of considering worst-case outcomes in robust optimization can be used to generalize the concepts of these risk measures. For example, we develop a model for the worst-case CVaR based on partial moment information when the exact distributions of uncertainties are unknown. This result extends the worst-case VaR results of El Ghaoui et al. (2003).

(b) We show how incoherent risk measures can be made coherent based on information about the support of the distribution of uncertainties in the optimization problem. Although this observation can be obtained alternatively using Theorem 2.2 in Ruszczyński and Shapiro (2006), our goal is to provide an explicit representation for practitioners looking to apply this theory in portfolio selection problems. We use duality theory to construct specific uncertainty sets that map to coherent risk measures, and to address the issue of the computational tractability of the resulting problems. We also explore the validity of probability bounds on the constraints in doing so, which is important for practical applications.

(c) Computationally, we study the effect of modifying incoherent risk measures into coherent risk measures on both theoretical and realized portfolio performance. These findings have implications for efficient portfolio optimization under different measures of risk.

While completing this paper, we became aware of work by Bertsimas and Brown (2009) that relates robust linear optimization to coherent risk measures. Both our paper and Bertsimas and Brown's paper address relationships between uncertainty sets in robust optimization and risk measures in finance; however, the approaches and the contributions of the two papers are different. Bertsimas and Brown's focus is on providing guidelines for selecting uncertainty sets in robust linear optimization applications based on the risk preferences of the modeler, and specifying uncertainty sets formed from available realizations of the uncertain data in the problem. In particular, they use the representation theorem for coherent risk measures that relates to the supremum of the expected value function over a family of distributions, and show, for example, that a class of coherent risk measures called *comonotone* for a discretely distributed random vector leads to polyhedral uncertainty sets. In contrast, we use mainly duality techniques to relate polyhedral and conic uncertainty sets with coherent risk measures. More generally, our approach is to use uncertainty sets in robust optimization as a starting point for constructing risk

measures in finance, with the goal of better understanding structural relationships between uncertainty sets and risk measures and improving computational tractability. This approach leads, for example, to showing that a moment-cone-based uncertainty set using semidefinite programming can lead to the worst-case CVaR measure. Recent work by Ben-Tal et al. (2009) is in the same spirit.

The structure of this paper is as follows: In §2, we review some popular financial measures of risk and the notion of coherent risk measures. In §3, we draw a parallel between using the concept of risk measure in finance and handling optimization problems with uncertain inputs. In §4, we review the main concepts of robust optimization, and analyze financial risk measures in the context of robust counterpart risk measures. In §5, we link the notion of coherent risk measures to robust optimization uncertainty sets, propose a method for constructing coherent risk measures from uncertainty sets, and prove that the probability of constraint violation remains the same for the resulting coherent robust counterpart risk measures. We illustrate the technique with a numerical example. Section 6 contains concluding remarks.

2. Risk Measures in Finance

In this section, we review the idea of portfolio risk measures in finance, list some of the most commonly used risk measures, and discuss the concept of *coherent risk measures*.

Consider a random vector $\tilde{\mathbf{z}}$ that is defined on a probability space (Ω, \mathcal{F}, P) , where Ω is a set of “outcomes,” \mathcal{F} is a set of “events,” and $P: \mathcal{F} \rightarrow [0, 1]$ is a function that assigns probability to events. In portfolio optimization, $\tilde{\mathbf{z}}$ could denote the underlying factor values in a factor model for returns or the returns of the assets themselves. We restrict our attention to the space of portfolio returns defined as an affine combination of the random variables (includes constants):

$$\mathcal{V} \triangleq \{\tilde{v}: \exists (v_0, \mathbf{v}) \text{ such that } \tilde{v} = v_0 + \mathbf{v}'\tilde{\mathbf{z}}\}. \quad (1)$$

Throughout the paper, the random vector $\tilde{\mathbf{z}}$ is assumed to have a finite and strictly positive definite covariance matrix. Given two random variables in $\tilde{v}, \tilde{w} \in \mathcal{V}$, in which $\tilde{v} = v_0 + \mathbf{v}'\tilde{\mathbf{z}}$ and $\tilde{w} = w_0 + \mathbf{w}'\tilde{\mathbf{z}}$, we use the notation $\tilde{v} \geq \tilde{w}$ to represent statewise dominance, i.e., $v_0 + \mathbf{v}'\mathbf{z} \geq w_0 + \mathbf{w}'\mathbf{z}$ for all $\mathbf{z} \in \Omega$. Similarly, $\tilde{v} > \tilde{w}$ denotes strict statewise dominance, i.e., $v_0 + \mathbf{v}'\mathbf{z} > w_0 + \mathbf{w}'\mathbf{z}$ for all $\mathbf{z} \in \Omega$. A risk measure $\rho(\tilde{v}): \mathcal{V} \rightarrow \Re$ assigns a real value to the random variable $\tilde{v} \in \mathcal{V}$. The smaller the value of the risk, the more desirable the portfolio is.

As an example, suppose the random asset returns are given as $\mathbf{r} + \mathbf{A}\tilde{\mathbf{z}}$ in a linear factor model. By moving the mean returns to \mathbf{r} , we can assume without loss of generality that the factors $\tilde{\mathbf{z}}$ have zero means. Such factor models for returns are widely used in finance (see, for example, Litterman et al. 2003). Let the vector \mathbf{x} denote the asset

allocation weights that are chosen from a subset $\mathcal{X} \subseteq \mathfrak{R}^n$. The set \mathcal{X} could include constraints on the portfolio structure, such as

(a) $\mathbf{e}'\mathbf{x} = 1$ (\mathbf{e} is the vector of ones; the weights of all assets in the portfolio add up to one),

(b) $\bar{\mathbf{x}} \geq \mathbf{x} \geq \underline{\mathbf{x}}$ (upper- and lower-bound constraints), etc.
 The random portfolio return is then defined as

$$\mathbf{x}'\mathbf{r} + \mathbf{x}'\mathbf{A}\tilde{\mathbf{z}} \in \mathcal{V},$$

with $(v_0, \mathbf{v}) = (\mathbf{x}'\mathbf{r}, \mathbf{A}'\mathbf{x})$ in (1). The portfolio optimization problem is then to find the minimum risk portfolio in the set of feasible portfolios:

$$\begin{aligned} \min \quad & \rho(\mathbf{x}'\mathbf{r} + \mathbf{x}'\mathbf{A}\tilde{\mathbf{z}}) \\ \text{s.t.} \quad & \mathbf{x} \in \mathcal{X}. \end{aligned} \quad (2)$$

2.1. Examples of Risk Measures

Most generally, risk measures in finance can be divided into two main categories: moment based and quantile based. The roots of moment-based risk measures can be traced to classical economic utility theory, whereas quantile-based risk measures have arisen as a consequence of advances in the theory of stochastic dominance. In this subsection, we list three of the most commonly used risk measures: mean-standard deviation (or, equivalently, mean-variance), Value-at-Risk (VaR), and Conditional Value-at-Risk (CVaR).

Mean-Standard Deviation. The classical mean-standard deviation portfolio allocation approach employs the risk measure

$$\rho_\alpha(v_0 + \mathbf{v}'\tilde{\mathbf{z}}) \triangleq -E(v_0 + \mathbf{v}'\tilde{\mathbf{z}}) + \alpha\sigma(v_0 + \mathbf{v}'\tilde{\mathbf{z}}),$$

where $E(\cdot)$ and $\sigma(\cdot)$ are the expected value and the standard deviation of the random portfolio return, and α is a parameter associated with the level of the investor's risk aversion. The mean-standard deviation risk measure is an example of a moment-based portfolio risk measure—it incorporates information about the first and second moments of the distribution of returns. Higher moments of the distribution of returns have been suggested as well (Huang and Litzenberger 1988); however, such risk measures have not become as popular due to computational and estimation difficulties.

In contrast to moment-based risk measures, quantile-based risk measures are concerned with the probability or magnitude of losses. The advantage of the quantile-based approach to risk measurement is that asymmetry in the distribution of returns can be handled better.

Value-at-Risk (VaR). The most popular quantile-based risk measure is Value-at-Risk. VaR measures the worst portfolio loss that can be expected with some small probability ϵ (ϵ typically equals 1% or 5%). Mathematically, the $(1 - \epsilon)$ -VaR is defined as follows:

$$\text{VaR}_{1-\epsilon}(v_0 + \mathbf{v}'\tilde{\mathbf{z}}) \triangleq \min\{t: P(-v_0 - \mathbf{v}'\tilde{\mathbf{z}} \leq t) \geq 1 - \epsilon\}.$$

Computationally, optimization of VaR is difficult to handle unless the distribution of returns is assumed to be normal or lognormal (Duffie and Pan 1997, Jorion 2000). Heuristics for optimizing sample VaR have been proposed in Gaivoronski and Pflug (2005) and Larsen et al. (2002). Sample-based approximations with probabilistic guarantees for VaR have been analyzed in Campi and Calafiore (2005), de Farias and Van Roy (2004), and Erdogan and Iyengar (2006). El Ghaoui et al. (2003) suggested an approach that optimizes the worst-case VaR for given mean and covariance matrix of the asset returns.

Conditional Value-at-Risk (CVaR). In recent years, an alternative quantile-based measure of risk known as Conditional Value-at-Risk (CVaR) has been gaining ground due to its attractive computational properties (Rockafellar and Uryasev 2000, 2002). CVaR measures the expected loss if the loss is above a specified quantile. Mathematically, the CVaR formulation can be written as

$$\text{CVaR}_{1-\epsilon}(v_0 + \mathbf{v}'\tilde{\mathbf{z}}) \triangleq \min_a \left(a + \frac{1}{\epsilon} E(-v_0 - \mathbf{v}'\tilde{\mathbf{z}} - a)^+ \right).$$

It can be shown that $\text{VaR}_{1-\epsilon}(v_0 + \mathbf{v}'\tilde{\mathbf{z}}) \leq \text{CVaR}_{1-\epsilon}(v_0 + \mathbf{v}'\tilde{\mathbf{z}})$. Hence, CVaR is often used as a conservative approximation of VaR (Rockafellar and Uryasev 2002). Furthermore, CVaR possesses the desirable property that it is a coherent risk measure. We will review the concept of coherent risk measures in the following subsection.

For the risk measures described above, the parameter α (in the case of mean-standard deviation) and ϵ (in the case of VaR and CVaR) determines the risk averseness of the decision maker.

2.2. Coherent Risk Measures

The risk measure $\rho: \mathcal{V} \rightarrow \mathfrak{R}$ assigns a real value to each uncertain outcome $\tilde{v} \in \mathcal{V}$. By convention, $\rho(\tilde{v}) \leq 0$ implies that the risk associated with an uncertain outcome \tilde{v} is acceptable. A risk measure $\rho(\cdot)$ is *coherent* if it satisfies the following four axioms (Artzner et al. 1999):

Axioms of coherent risk measures:

- (i) Translation invariance: For all $\tilde{v} \in \mathcal{V}$ and $a \in \mathfrak{R}$, $\rho(\tilde{v} + a) = \rho(\tilde{v}) - a$.
- (ii) Subadditivity: For all random variables $\tilde{v}, \tilde{w} \in \mathcal{V}$, $\rho(\tilde{v} + \tilde{w}) \leq \rho(\tilde{v}) + \rho(\tilde{w})$.
- (iii) Positive homogeneity: For all $\tilde{v} \in \mathcal{V}$ and $\lambda \geq 0$, $\rho(\lambda\tilde{v}) = \lambda\rho(\tilde{v})$.
- (iv) Monotonicity: For all random variables $\tilde{v}, \tilde{w} \in \mathcal{V}$ such that $\tilde{v} \geq \tilde{w}$, $\rho(\tilde{v}) \leq \rho(\tilde{w})$.

In addition, we define a coherent risk measure as *proper* if it satisfies the following condition:

- For all $\tilde{v} \in \mathcal{V}$ with positive variance, $\rho(\tilde{v}) > E(-\tilde{v})$.

A proper coherent risk measure ensures that the risk of an asset with random returns is always greater than its risk-neutral value.

One important consequence of the coherent risk measure axioms is preservation of convexity, which is important for computational tractability in portfolio optimization (Ruszczynski and Shapiro 2006).

3. Risk Measures and Optimization Under Uncertainty

The framework of risk measures in portfolio optimization described in the previous section can be extended to more general optimization problems with parameter uncertainties. Consider a linear optimization problem:

$$\begin{aligned} \min \quad & \mathbf{c}'\mathbf{x} \\ \text{s.t.} \quad & \mathbf{a}_i(\mathbf{z})'\mathbf{x} \geq b_i(\mathbf{z}), \quad i \in I, \end{aligned} \tag{3}$$

where \mathbf{x} denotes the vector of decision variables; $\mathbf{a}_i(\mathbf{z})$, $b_i(\mathbf{z})$, $i \in I$ are data that depends affinely on the factors \mathbf{z} . Without loss of generality, we assume that \mathbf{c} is known exactly, and the uncertainty affects only parameters \mathbf{z} . For any fixed solution \mathbf{x} , the constraint $\mathbf{a}_i(\mathbf{z})'\mathbf{x} \geq b_i(\mathbf{z})$ may become infeasible for some realization of \mathbf{z} . In many applications of optimization, ensuring constraint feasibility for all realization of uncertainties can be overly conservative. In such problems, we can tolerate some risk of constraint violation for the benefit of improving the objective. Therefore, it is natural to extend the optimization framework using risk constraints as follows:

$$\begin{aligned} \min \quad & \mathbf{c}'\mathbf{x} \\ \text{s.t.} \quad & \rho_i(\mathbf{a}_i(\tilde{\mathbf{z}})'\mathbf{x} - b_i(\tilde{\mathbf{z}})) \leq 0, \quad i \in I, \end{aligned} \tag{4}$$

with $\mathbf{a}_i(\tilde{\mathbf{z}})'\mathbf{x} - b_i(\tilde{\mathbf{z}}) \in \mathcal{V}$ for $i \in I$. Model (4) is called the *risk counterpart* of (3). In particular, if we define risk as the probability that the constraint violation will be more than a prespecified constant ϵ (i.e., if we introduce chance constraints as in Charnes and Cooper 1959), formulation (4) is equivalent to VaR optimization.

Even if the nominal problem without uncertainty is computationally tractable, the choice of the risk measure can affect the tractability of the risk counterpart. Under the VaR risk measure, the risk counterpart can become nonconvex and intractable. Because coherent risk measures preserve convexity, risk counterparts with the CVaR measure are generally easier to optimize than with VaR. Although convexity is maintained, an important consideration with regard to tractability is also whether a risk measure can be computed to an arbitrary accuracy. This is essential when an optimization problem contains constraints that need to be satisfied with high reliability, such as in the case of structural design (see Ben-Tal and Nemirovski 2001). For example, the computation of $\text{CVaR}_{1-\epsilon}(v_0 + \mathbf{v}'\tilde{\mathbf{z}})$ involves multidimensional integration, which is a computationally challenging task. Although the integrals can be approximated through Monte Carlo simulation, the number of trials to achieve high reliability can be prohibitive. At the same time, if first- and second-moment information about the distribution of the uncertainties is available, the mean-standard deviation risk measure has better computational characteristics despite the fact that it is not a coherent risk measure.

4. Uncertainty Sets in Robust Optimization

In practice, the exact distributions of uncertain parameters in optimization models are rarely known. Robust optimization handles this issue by requiring the user to specify a (deterministic) uncertainty set for the parameters based on some (possibly limited) information about their values. The key idea is then to find an optimal solution to the problem that remains feasible for any realization of the uncertain coefficients within the prespecified deterministic uncertainty set. The *robust counterpart* analogous to the portfolio optimization problem (2) with risk measures is then formulated as

$$\begin{aligned} \min \quad & \left(\max_{\mathbf{z} \in \mathcal{U}} -(\mathbf{x}'\mathbf{r} + \mathbf{x}'\mathbf{A}\mathbf{z}) \right) \\ \text{s.t.} \quad & \mathbf{x} \in X, \end{aligned} \tag{5}$$

where $-(\mathbf{x}'\mathbf{r} + \mathbf{x}'\mathbf{A}\mathbf{z})$ is the loss of the portfolio and \mathcal{U} is a deterministic uncertainty set that is mapped out from the uncertain factors independent of \mathbf{x} . Typically, the uncertainty set is convex, and its size is related to some kind of guarantee on the probability that the constraint involving the uncertain data will not be violated (see El Ghaoui et al. 1998, 2003; Ben-Tal and Nemirovski 2001; Bertsimas and Sim 2004; Bertsimas et al. 2004; Chen et al. 2007).

In view of (2) and (5), we define the *robust counterpart risk measure* for any $\tilde{v} = v_0 + \mathbf{v}'\tilde{\mathbf{z}} \in \mathcal{V}$ as

$$\eta_{\mathcal{U}}(v_0 + \mathbf{v}'\tilde{\mathbf{z}}) \triangleq \max_{\mathbf{z} \in \mathcal{U}} -(v_0 + \mathbf{v}'\mathbf{z}) = -\min_{\mathbf{z} \in \mathcal{U}} (v_0 + \mathbf{v}'\mathbf{z}). \tag{6}$$

In line with the convention for risk measures, $\eta_{\mathcal{U}}(v_0 + \mathbf{v}'\tilde{\mathbf{z}}) \leq 0$ implies that the risk associated with the violation of the uncertain constraint, $v_0 + \mathbf{v}'\tilde{\mathbf{z}} \geq 0$, is acceptable. Therefore, one can think of the definition of an uncertainty set as the definition of a risk measure on the uncertainties involved.

4.1. Examples of Uncertainty Sets and Corresponding Risk Measures

We next illustrate with examples the correspondence between risk measures and robust optimization uncertainty sets. These examples show also that robust optimization uncertainty sets can be used to generalize the definitions of risk measures in finance.

Ellipsoidal Uncertainty Set. One of the most commonly used uncertainty sets in robust optimization is the ellipsoidal uncertainty set. Assume that the primitive uncertainties lie in the ellipsoidal uncertainty set given as

$$\mathcal{E}_{\alpha} \triangleq \{\mathbf{z}: \|\mathbf{Q}^{-1/2}\mathbf{z}\|_2 \leq \alpha\}.$$

The robust counterpart risk measure is then defined as

$$\eta_{\mathcal{E}_{\alpha}}(v_0 + \mathbf{v}'\tilde{\mathbf{z}}) \triangleq -\min_{\mathbf{z} \in \mathcal{E}_{\alpha}} (v_0 + \mathbf{v}'\mathbf{z}).$$

This corresponds to minimizing an affine function over a single ellipsoidal constraint and is solvable in closed form (see Ben-Tal and Nemirovski 2001). The robust counterpart risk measure is equivalent to

$$\eta_{\mathcal{E}_\alpha}(v_0 + \mathbf{v}'\tilde{\mathbf{z}}) = -v_0 + \alpha\sqrt{\mathbf{v}'\mathbf{Q}\mathbf{v}}, \quad (7)$$

where the factors $\tilde{\mathbf{z}}$ are assumed to have zero means and covariance matrix \mathbf{Q} . Clearly, the ellipsoidal uncertainty set maps to the mean-standard deviation portfolio risk measure discussed in §2.1.

The ellipsoidal uncertainty set also arises in the worst-case $(1 - \epsilon)$ -VaR risk measure discussed in El Ghaoui et al. (2003). Suppose the random factors $\tilde{\mathbf{z}}$ have zero mean and covariance matrix \mathbf{Q} , but the exact distribution is unknown. Let \mathcal{P} denote the set of all possible probability distributions with the given mean and covariance matrix. The worst-case $(1 - \epsilon)$ -VaR is then defined as

$$\text{wcVaR}_{1-\epsilon}(v_0 + \mathbf{v}'\tilde{\mathbf{z}}) \triangleq \min \left\{ t : \inf_{P \in \mathcal{P}} P(-v_0 - \mathbf{v}'\tilde{\mathbf{z}} \leq t) \geq 1 - \epsilon \right\}.$$

As shown in El Ghaoui et al. (2003), this reduces to the mean-standard deviation risk measure with $\alpha = \sqrt{(1 - \epsilon)/\epsilon}$:

$$\text{wcVaR}_{1-\epsilon}(v_0 + \mathbf{v}'\tilde{\mathbf{z}}) = \eta_{\mathcal{E}_{\sqrt{(1-\epsilon)/\epsilon}}}(v_0 + \mathbf{v}'\tilde{\mathbf{z}}).$$

Polyhedral Uncertainty Set. Another commonly used uncertainty set in robust linear optimization is the polyhedral uncertainty set. We show the connection between a particular data-driven example of this uncertainty set and CVaR for a given discrete distribution.

THEOREM 1. *Consider a random vector $\tilde{\mathbf{z}}$ with the discrete distribution $P(\tilde{\mathbf{z}} = \mathbf{z}_k) = p_k, k = 1, \dots, M$. The robust counterpart risk measure for the uncertainty set*

$$\mathcal{U}_{1-\epsilon} = \left\{ \mathbf{z} : \begin{array}{l} \exists \mathbf{u} \in \mathfrak{N}^M \\ \mathbf{z} = \sum_{k=1}^M u_k \mathbf{z}_k, \\ \sum_{k=1}^M u_k = 1, \\ \mathbf{0} \leq \mathbf{u} \leq \frac{1}{\epsilon} \mathbf{p}, \end{array} \right.$$

is the conditional value risk measure

$$\text{CVaR}_{1-\epsilon}(v_0 + \mathbf{v}'\tilde{\mathbf{z}}) = \eta_{\mathcal{U}_{1-\epsilon}}(v_0 + \mathbf{v}'\tilde{\mathbf{z}}),$$

PROOF. The equivalent representation of the $(1 - \epsilon)$ -CVaR is

$$\begin{aligned} \text{CVaR}_{1-\epsilon}(v_0 + \mathbf{v}'\tilde{\mathbf{z}}) &= \min_{a, \mathbf{y}} a + \frac{1}{\epsilon} \sum_{k=1}^M p_k y_k \\ \text{s.t. } a + y_k &\geq -v_0 - \mathbf{v}'\mathbf{z}_k, \\ &k = 1, \dots, M, \\ \mathbf{y} &\geq \mathbf{0}. \end{aligned}$$

Using strong duality from linear programming,

$$\begin{aligned} \text{CVaR}_{1-\epsilon}(v_0 + \mathbf{v}'\tilde{\mathbf{z}}) &= \max_{\mathbf{u}} - \sum_{k=1}^M u_k (v_0 + \mathbf{v}'\mathbf{z}_k) \\ \text{s.t. } \sum_{k=1}^M u_k &= 1, \\ \mathbf{0} \leq \mathbf{u} &\leq \frac{1}{\epsilon} \mathbf{p}, \end{aligned}$$

or equivalently,

$$\begin{aligned} \text{CVaR}_{1-\epsilon}(v_0 + \mathbf{v}'\tilde{\mathbf{z}}) &= - \min_{\mathbf{u}, \mathbf{z}} (v_0 + \mathbf{v}'\mathbf{z}) \\ \text{s.t. } \mathbf{z} &= \sum_{k=1}^M u_k \mathbf{z}_k, \\ \sum_{k=1}^M u_k &= 1, \\ \mathbf{0} \leq \mathbf{u} &\leq \frac{1}{\epsilon} \mathbf{p}. \end{aligned}$$

This clearly yields the desired uncertainty set. \square

As mentioned earlier, Bertsimas and Brown (2009) provide a generalized result about the relationship between data-driven polyhedral uncertainty sets and so-called *comonotone* risk measures in finance. They show that the entire space of such polyhedral uncertainty sets can be finitely generated by the class of CVaR risk measures.

Moment Cone Uncertainty Set. We now show the equivalence between a specific moment cone uncertainty set and the worst-case CVaR risk measure. Suppose the probability measure P of the random factors $\tilde{\mathbf{z}}$ is not exactly known; rather, it is known to lie in a set $P \in \mathcal{P}$. It is natural in this setting to define the worst-case $(1 - \epsilon)$ -CVaR risk measure as

$$\text{wcCVaR}_{1-\epsilon}(v_0 + \mathbf{v}'\tilde{\mathbf{z}}) \triangleq \sup_{P \in \mathcal{P}} \text{CVaR}_{1-\epsilon}(v_0 + \mathbf{v}'\tilde{\mathbf{z}}). \quad (8)$$

Tractable formulations for specific choices of \mathcal{P} have been obtained in Zhu and Fukushima (2009) and Čerbačková (2005). We use a more general moments approach to characterize the set of distributions \mathcal{P} and derive the corresponding uncertainty set. Let

$$\mathcal{J}_d \triangleq \{\boldsymbol{\beta} \in \mathfrak{N}^m : \beta_1 + \dots + \beta_m \leq d\}$$

be an index set indexed by $\boldsymbol{\beta} = (\beta_1, \dots, \beta_m)$ that defines the set of monomials of degree less than or equal to d in the variables $\tilde{\mathbf{z}} = (\tilde{z}_1, \dots, \tilde{z}_m)$. Suppose we are given the moments $\mathbf{m} \in \mathfrak{N}^{|\mathcal{J}_d|}$. Let $\mathbb{M}(\Omega)$ denote the set of finite positive Borel measures supported on a compact set Ω . We define the set of feasible probability measures as

$$\mathcal{P} = \{P \in \mathbb{M}(\Omega) : E_P(\tilde{\mathbf{z}}^{\boldsymbol{\beta}}) = m_{\boldsymbol{\beta}} \forall \boldsymbol{\beta} \in \mathcal{J}_d\}, \quad (9)$$

where $\tilde{\mathbf{z}}^\beta = \tilde{z}_1^{\beta_1} \times \dots \times \tilde{z}_m^{\beta_m}$. We explicitly include $\beta = \mathbf{0}$ and $m_0 = 1$ in (9) to ensure that the measures are probability measures. A simple model could include mean, variance, covariance, and range information on $\tilde{\mathbf{z}}$. Note that no explicit assumption on independence is made, thus naturally extending the multidimensional model of CVaR to dependent distributions.

For $\Omega \subseteq \mathfrak{R}^m$, let the cone of moments supported on Ω be defined as

$$\mathbb{M}_{m,d}(\Omega) = \left\{ \mathbf{w} \in \mathfrak{R}^{|\mathcal{J}_d|} : \mathbf{w}_\beta = E_P(\tilde{\mathbf{z}}^\beta) \quad \forall \beta \in \mathcal{J}_d \text{ for some } P \in \mathbb{M}(\Omega) \right\},$$

and the cone of positive polynomials supported on Ω be defined as

$$\mathbb{P}_{m,d}(\Omega) = \left\{ \mathbf{y} \in \mathfrak{R}^{|\mathcal{J}_d|} : \sum_{\beta \in \mathcal{J}_d} y_\beta z^\beta \geq 0 \quad \forall \mathbf{z} \in \Omega \right\},$$

where $\mathbf{y} = (y_\beta)_{\beta \in \mathcal{J}_d}$ are the coefficients of the polynomial. These cones are dual to each other (Karlin and Studden 1966); namely, the closure of the moment cone is precisely the dual cone of the set of nonnegative polynomials on Ω :

$$\overline{\mathbb{M}_{m,d}(\Omega)} = \mathbb{P}_{m,d}^*(\Omega).$$

We now show the connection between the moment cone uncertainty set and the worst-case CVaR risk measure.

THEOREM 2. *Consider a random vector $\tilde{\mathbf{z}}$ with known moments $\mathbf{m} \in \overline{\mathbb{M}_{m,d}(\Omega)}$. Assume that Ω is compact. Then, the robust counterpart risk measure for the uncertainty set*

$$\mathcal{U}_{1-\epsilon} = \left\{ \begin{array}{l} \exists \mathbf{w}, \mathbf{s} \in \mathfrak{R}^{|\mathcal{J}_d|} \\ z_j = w_{\mathbf{e}_j}, \quad j = 1, \dots, m, \\ \mathbf{z}: \mathbf{w} + \mathbf{s} = \frac{1}{\epsilon} \mathbf{m}, \\ \mathbf{w}, \mathbf{s} \in \overline{\mathbb{M}_{m,d}(\Omega)} \\ w_0 = 1, \end{array} \right.$$

is the worst-case CVaR risk measure

$$\text{wcCVaR}_{1-\epsilon}(v_0 + \mathbf{v}'\tilde{\mathbf{z}}) = \eta_{\mathcal{U}_{1-\epsilon}}(v_0 + \mathbf{v}'\tilde{\mathbf{z}}),$$

where \mathbf{e}_j is an m -dimensional vector with one in the j th component and zero otherwise.

PROOF. Using the result from Rockafellar and Uryasev (2002), the worst-case $(1 - \epsilon)$ -CVaR can be expressed as

$$\begin{aligned} & \text{wcCVaR}_{1-\epsilon}(v_0 + \mathbf{v}'\tilde{\mathbf{z}}) \\ &= \sup_{P \in \mathcal{P}} \inf_{a \in \mathfrak{R}} \left(a + \frac{1}{\epsilon} E_P(-v_0 - \mathbf{v}'\tilde{\mathbf{z}} - a)^+ \right). \end{aligned} \tag{10}$$

By changing the order of the supremum and infimum in (10), we have

$$\text{wcCVaR}_{1-\epsilon}(v_0 + \mathbf{v}'\tilde{\mathbf{z}}) \leq \inf_{a \in \mathfrak{R}} \left(a + \frac{1}{\epsilon} \sup_{P \in \mathcal{P}} E_P(-v_0 - \mathbf{v}'\tilde{\mathbf{z}} - a)^+ \right).$$

First, note that the objective function is linear in the probability measure P and convex in decision variable a . Furthermore, the value of a is guaranteed to lie in a bounded interval for a fixed $\epsilon \in (0, 1)$. Hence, the min-max and the max-min problem coincide in the optimal value (see Shapiro and Kleywegt 2002). Under strong duality conditions for the moments problem, we have

$$\begin{aligned} \sup_{P \in \mathcal{P}} E_P(-v_0 - \mathbf{v}'\tilde{\mathbf{z}} - a)^+ &= \inf_{\mathbf{y}} \mathbf{m}'\mathbf{y} \\ \text{s.t. } & \mathbf{y} + (a + v_0, \mathbf{v}', \mathbf{0})' \in \mathbb{P}_{m,d}(\Omega), \\ & \mathbf{y} \in \mathbb{P}_{m,d}(\Omega). \end{aligned}$$

Substituting this dual formulation for the inner problem into the worst-case CVaR problem, we obtain

$$\begin{aligned} \text{wcCVaR}_{1-\epsilon}(v_0 + \mathbf{v}'\tilde{\mathbf{z}}) &= \inf_{a, \mathbf{y}} \left(a + \frac{1}{\epsilon} \mathbf{m}'\mathbf{y} \right) \\ \text{s.t. } & \mathbf{y} + (a + v_0, \mathbf{v}', \mathbf{0})' \in \mathbb{P}_{m,d}(\Omega), \\ & \mathbf{y} \in \mathbb{P}_{m,d}(\Omega). \end{aligned}$$

Dualizing this formulation, we obtain the primal moments formulation:

$$\begin{aligned} \text{wcCVaR}_{1-\epsilon}(v_0 + \mathbf{v}'\tilde{\mathbf{z}}) &= \sup_{\mathbf{w}, \mathbf{s}} -v_0 - \mathbf{v}'\mathbf{z} \\ \text{s.t. } & z_j = w_{\mathbf{e}_j}, \quad j = 1, \dots, m, \\ & \mathbf{w} + \mathbf{s} = \frac{1}{\epsilon} \mathbf{m}, \\ & \mathbf{w}, \mathbf{s} \in \overline{\mathbb{M}_{m,d}(\Omega)}, \\ & w_0 = 1, \end{aligned} \tag{11}$$

which yields the desired result. \square

For a fairly large class of Ω , membership in the moment cone $\overline{\mathbb{M}_{m,d}(\Omega)}$ can either be represented exactly or approximated using semidefinite constraints (Lasserre 2002, Zuluaga and Pena 2005). Examples for which the representation is exact and polynomial sized in the dimension of the problem include:

- (a) Discrete support $\Omega = \{\mathbf{z}_1, \mathbf{z}_2, \dots, \mathbf{z}_M\}$. In this case, the representation simply reduces to linear constraints.
- (b) $\Omega = \prod_{i=1}^n [a_i, b_i]$ with a finite set of known marginal moments. In this case, the representation reduces to semidefinite constraints (see Zuluaga and Pena 2005 and Bertsimas et al. 2006).

More generally, the membership constraint can be represented asymptotically by using larger and larger semidefinite relaxations (Lasserre 2002). Examples of such Ω include:

- (a) A bounded polyhedron.
- (b) Compact set with a bound B known a priori such that $\Omega \subseteq \{\mathbf{z}: \mathbf{z}'\mathbf{z} \leq B^2\}$.

Theorem 2 generalizes the idea of worst-case VaR introduced by El Ghaoui et al. (2003) to worst-case CVaR. It should be noted that while extending the former notion to higher-order moments is not easy (due to the nonconvexity of the formulation), it seems easier to obtain stronger approximations for worst-case CVaR.

Moment-Generating Function Uncertainty Set. We next consider an uncertainty set that is implicitly defined from the moment-generating functions of the uncertain return factors $\tilde{\mathbf{z}}$. We assume that \tilde{z}_j , $j = 1, \dots, m$ are stochastically independent and their moment-generation functions, $g_j(\theta) \triangleq E(\exp(\theta\tilde{z}_j))$, are well defined. Motivated by Nemirovski and Shapiro's (2006) approximation on the chance-constrained problem, we define the risk measure

$$\begin{aligned} \zeta_{1-\epsilon}(v_0 + \mathbf{v}'\tilde{\mathbf{z}}) &\triangleq \inf_{\mu > 0} (\mu \ln(E(\exp(-(v_0 + \mathbf{v}'\tilde{\mathbf{z}})/\mu))) - \mu \ln \epsilon) \\ &= \inf_{\mu > 0} \left(-v_0 + \mu \sum_{j=1}^m \ln(g_j(-v_j/\mu)) - \mu \ln \epsilon \right). \end{aligned}$$

Nemirovski and Shapiro (2006) show that the risk measure is computationally tractable if the moment-generating functions $g_j(\theta)$ are computationally tractable functions, which is indeed the case for common distributions. Moreover, they show that $\zeta_{1-\epsilon}(\cdot)$ is an upper bound of the $(1 - \epsilon)$ -CVaR risk measure. The key idea of bound comes from the observation that

$$w^+ \leq \frac{\mu}{e} \exp(w/\mu) \quad \forall \mu > 0.$$

Hence,

$$\begin{aligned} \text{CVaR}_{1-\epsilon}(v_0 + \mathbf{v}'\tilde{\mathbf{z}}) &= \inf_a \left(a + \frac{1}{\epsilon} E(-v_0 - \mathbf{v}'\tilde{\mathbf{z}} - a)^+ \right) \\ &\leq \inf_{a, \mu > 0} \left(a + \frac{\mu}{e\epsilon} E(\exp((-a - v_0 - \mathbf{v}'\tilde{\mathbf{z}})/\mu)) \right) \\ &= \inf_{\mu > 0} (\mu \ln E(\exp(-(v_0 + \mathbf{v}'\tilde{\mathbf{z}})/\mu))) - \mu \ln \epsilon \\ &= \zeta_{1-\epsilon}(v_0 + \mathbf{v}'\tilde{\mathbf{z}}), \end{aligned}$$

where the second equality follows from choosing the minimizers a^* as follows:

$$a^* = \mu \ln E(\exp(-(v_0 + \mathbf{v}'\tilde{\mathbf{z}})/\mu)) - \mu - \mu \ln \epsilon.$$

Consequently,

$$\text{VaR}_{1-\epsilon}(v_0 + \mathbf{v}'\tilde{\mathbf{z}}) \leq \text{CVaR}_{1-\epsilon}(v_0 + \mathbf{v}'\tilde{\mathbf{z}}) \leq \zeta_{1-\epsilon}(v_0 + \mathbf{v}'\tilde{\mathbf{z}}).$$

Under the assumption that $\tilde{\mathbf{z}}$ are independently distributed and that their moment-generating functions are computationally tractable functions, the risk measure $\zeta_{1-\epsilon}(\cdot)$ is convex and is computationally tractable. However, it is a weaker approximation of the VaR measure compared to the CVaR measure.

THEOREM 3. Consider a random vector $\tilde{\mathbf{z}}$ in the sample space Ω in which the elements are independently distributed and their moment-generating functions are computationally tractable. Define the uncertainty set

$$\mathcal{B}_{1-\epsilon} = \{\mathbf{z}: (-\mathbf{z}, 0, 1) \in \Psi_{1-\epsilon}^*\},$$

where $\Psi_{1-\epsilon}^*$ is the dual cone of

$$\Psi_{1-\epsilon} = \text{cl} \left(\left\{ (\mathbf{y}, \mu, t): \mu \sum_{j=1}^m \ln \left(g_j \left(\frac{y_j}{\mu} \right) \right) - \mu \ln \epsilon \leq t, \mu > 0 \right\} \right),$$

in which $\text{cl}(\cdot)$ denotes closure of the set. Then,

$$\zeta_{1-\epsilon}(v_0 + \mathbf{v}'\tilde{\mathbf{z}}) = \eta_{\mathcal{B}_{1-\epsilon}}(v_0 + \mathbf{v}'\tilde{\mathbf{z}}).$$

Moreover, $\mathcal{B}_{1-\epsilon} \subseteq \mathcal{CH}(\Omega)$, where $\mathcal{CH}(\Omega)$ represents the convex hull of Ω .

PROOF. As noted by Nemirovski and Shapiro (2006), the function

$$\phi_\epsilon(\mathbf{y}, \mu) = \mu \sum_{j=1}^m \ln \left(g_j \left(\frac{y_j}{\mu} \right) \right) - \mu \ln \epsilon$$

is jointly convex in \mathbf{y} and $\mu > 0$. Therefore, the cone $\Psi_{1-\epsilon}$ is convex, closed with a nonempty interior. Furthermore, because \tilde{z}_j has zero mean, for nonzero random variable \tilde{z}_j we have $g_j(y) \rightarrow \infty$ if and only if $|y| \rightarrow \infty$. Hence, the cone is pointed as well. Therefore, the dual cone, Ψ_ϵ^* is also a closed, pointed convex cone with nonempty interior (see Rockafellar 1970). Observe that we have

$$\begin{aligned} \zeta_{1-\epsilon}(v_0 + \mathbf{v}'\tilde{\mathbf{z}}) &= \inf_{\mu > 0} (\mu \ln(E(\exp(-(v_0 + \mathbf{v}'\tilde{\mathbf{z}})/\mu))) - \mu \ln \epsilon) \\ &= \min_{\mu, t} (t: (-\mathbf{v}, \mu, t + v_0) \in \Psi_{1-\epsilon}) \\ &= \max_{\mathbf{u}, p, s} (-v_0 s + \mathbf{v}'\mathbf{u}: p = 0, s = 1, (\mathbf{u}, p, s) \in \Psi_{1-\epsilon}^*) \\ &= - \min_{\mathbf{z} \in \mathcal{B}_{1-\epsilon}} (v_0 + \mathbf{v}'\mathbf{z}), \end{aligned}$$

where the second last equality follows from strong conic duality because the primal problem is bounded and Slater's conditions are satisfied in the primal problem. The reader is referred to Chapter 2 of Ben-Tal and Nemirovski (2001) for details on conic duality.

Finally, to prove that $\mathcal{B}_{1-\epsilon} \subseteq \mathcal{CH}(\Omega)$, it suffices to show that for all \mathbf{v} ,

$$\min_{\mathbf{z} \in \mathcal{B}_{1-\epsilon}} \mathbf{v}'\mathbf{z} \geq \min_{\mathbf{z} \in \mathcal{CH}(\Omega)} \mathbf{v}'\mathbf{z}.$$

Indeed,

$$\begin{aligned} - \min_{\mathbf{z} \in \mathcal{B}_{1-\epsilon}} \mathbf{v}'\mathbf{z} &= \inf_{\mu > 0} (\mu \ln(E(\exp(-\mathbf{v}'\tilde{\mathbf{z}})/\mu))) - \mu \ln \epsilon \\ &\leq \inf_{\mu > 0} \left(\mu \ln \left(\exp \left(\max_{\mathbf{z} \in \mathcal{CH}(\Omega)} (-\mathbf{v}'\mathbf{z}) / \mu \right) \right) - \mu \ln \epsilon \right) \\ &= - \min_{\mathbf{z} \in \mathcal{CH}(\Omega)} \mathbf{v}'\mathbf{z}. \quad \square \end{aligned}$$

The risk measure $\zeta_{1-\epsilon}(\cdot)$ is also a coherent risk measure. This fact follows immediately from Theorem 4, which we prove in the next section.

5. Coherent Risk Measures and Uncertainty Sets

In this section, we propose a method for constructing coherent risk measures based on robust optimization uncertainty sets with support information, and derive bounds on the probability of constraint violation under the so-constructed risk measures. We illustrate the method with a numerical example.

5.1. Creating Coherent Risk Measures

It is well known that one can describe any coherent risk measure equivalently in terms of the worst-case expectation over a family of distributions, \mathcal{P} , as follows:

$$\rho(\tilde{v}) = \sup_{P \in \mathcal{P}} E_P(-\tilde{v}) \tag{12}$$

(see, for example, Theorem 2.2 in Ruszczyński and Shapiro 2006, or Proposition 3.1 in Föllmer and Schied 2002). The representation result (12), however, does not always imply that the problem of minimizing coherent risk measures in portfolio optimization is of polynomial complexity. To address this issue, we show in Theorem 4 that we can describe any proper coherent risk measure defined on the uncertain portfolio returns in terms of the worst-case return over a deterministic uncertainty set. This result can also be derived from well-known results (see Theorem 2.2 in Ruszczyński and Shapiro 2006). Our goal, however, is to specify explicit uncertainty set construction for such risk measures. In Theorem 5, we provide a class of uncertainty sets for which the minimization of the corresponding coherent risk measure can be carried out in polynomial time.

THEOREM 4. *Consider a random vector $\tilde{\mathbf{z}}$ in the sample space Ω , for which $E(\tilde{\mathbf{z}}) = \mathbf{0}$, and whose covariance matrix is strictly positive definite. A risk measure $\rho(v_0 + \mathbf{v}'\tilde{\mathbf{z}})$ defined on \mathcal{V} is a proper coherent risk measure if and only if*

$$\rho(v_0 + \mathbf{v}'\tilde{\mathbf{z}}) = \eta_{\mathcal{C}}(v_0 + \mathbf{v}'\tilde{\mathbf{z}}),$$

for some convex uncertainty set \mathcal{C} with $\mathbf{0}$ in the interior and $\mathcal{C} \subseteq \mathcal{CH}(\Omega)$. In particular, the uncertainty set \mathcal{C} associated with the risk measure $\rho(\cdot)$ is given by

$$\mathcal{C} = \left\{ \mathbf{z}: \max_{\mathbf{y}} \{-\mathbf{z}'\mathbf{y}: \rho(\mathbf{y}'\tilde{\mathbf{z}}) \leq 1\} \leq 1 \right\}.$$

PROOF. The proof is based on conic duality and is provided in the online appendix. \square

An electronic companion to this paper is available as part of the online version that can be found at <http://or.journal.informs.org/>.

REMARK. Based on the representation of coherent risk measure in (12), it is straightforward to show that $\rho(v_0 + \mathbf{v}'\tilde{\mathbf{z}}) = \eta_{\mathcal{C}}(v_0 + \mathbf{v}'\tilde{\mathbf{z}})$ for some uncertainty set \mathcal{C} given by

$$\mathcal{C} = \mathcal{CH}(\{\mathbf{z}: \mathbf{z} = E_P(\tilde{\mathbf{z}}), P \in \mathcal{P}\}),$$

where the family of distributions \mathcal{P} is given as a “black box.” However, given the space of random variables \mathcal{V} we consider, we are able to obtain an explicit representation of the uncertainty set \mathcal{C} using the risk measure ρ . For example, the uncertainty set for the mean-standard deviation risk measure given in (7) can be obtained as follows:

$$\begin{aligned} \mathcal{C} &= \left\{ \mathbf{z}: \max_{\mathbf{y}} \{-\mathbf{z}'\mathbf{y}: \rho(\mathbf{y}'\tilde{\mathbf{z}}) \leq 1\} \leq 1 \right\} \\ &= \left\{ \mathbf{z}: \max_{\mathbf{y}} \{-\mathbf{z}'\mathbf{y}: \alpha \|\mathbf{Q}^{1/2}\mathbf{y}\|_2 \leq 1\} \leq 1 \right\} \\ &= \left\{ \mathbf{z}: 1/\alpha \max_{\mathbf{y}} \{\mathbf{z}'\mathbf{y}: \|\mathbf{Q}^{1/2}(-\mathbf{y})\|_2 \leq 1\} \leq 1 \right\} \\ &= \left\{ \mathbf{z}: \max_{\mathbf{y}} \{\mathbf{z}'\mathbf{y}: \|\mathbf{Q}^{1/2}\mathbf{y}\|_2 \leq 1\} \leq \alpha \right\} \\ &= \left\{ \mathbf{z}: \max_{\mathbf{y}} \{(\mathbf{Q}^{-1/2}\mathbf{z})'\mathbf{y}: \|\mathbf{y}\|_2 \leq 1\} \leq \alpha \right\} \\ &= \{\mathbf{z}: \|\mathbf{Q}^{-1/2}\mathbf{z}\|_2 \leq \alpha\} = \mathcal{E}_{\alpha}, \end{aligned}$$

which follows from the self-dual property of Euclidean norm, $\|\mathbf{a}\|_2 = \max_{\mathbf{y}: \|\mathbf{y}\|_2 \leq 1} \mathbf{a}'\mathbf{y}$. Hence, Theorem 4 also implies that the mean-standard deviation risk measure is coherent if and only if $\mathcal{E}_{\alpha} \subseteq \mathcal{CH}(\Omega)$. This relationship is not obvious from (12).

From Theorem 4 and the remark above, it is clear that given any uncertainty set \mathcal{W} that is not necessarily a subset of $\mathcal{CH}(\Omega)$, we can make the associated risk measure a coherent one by modifying the uncertainty set to

$$\mathcal{U} = \mathcal{W} \cap \bar{\Omega},$$

where $\bar{\Omega} \subseteq \mathcal{CH}(\Omega)$.

Although it is obvious that a decision maker would not try to protect against realizations of the uncertain parameters that do not lie in their support set, classical uncertainty sets used in robust optimization do not in fact always satisfy this condition. For example, specifying an uncertainty set that relies on a nominal estimate plus or minus three standard deviations of the uncertain parameter may “over protect” on one side if the distribution of the uncertain parameter is asymmetric, and thus extend beyond the support set for the uncertain parameter. Including support information thus becomes an important consideration in practice.

Defining an uncertainty set through Theorem 4 has some advantages. First, it is implicitly defined through the proper coherent risk measure, instead of using conic quadratic constraints on the uncertain factors, which is ubiquitous in the area of applied robust optimization. Second, Theorem 4 shows that to construct a coherent risk measure, we only need to require the convex hull of the sample space, which can lead to computationally friendly formulations. In contrast, it may be computationally expensive to construct a risk measure based on a set of worst-case expectations over a family of distributions defined on the sample space, due to the possibly exponentially large number of scenarios. For example, if $\tilde{z}_1, \dots, \tilde{z}_m$ are independently distributed

Bernoulli random variables in $[-1, 1]$, the sample space comprises 2^m scenarios. At the same time, its convex hull is a hypercube, which can be concisely represented as the intersection of a small number of hyperplanes.

More generally, assume that the uncertainty sets are conic representable,

$$\mathcal{U} = \{\mathbf{z}: \mathbf{D}\mathbf{z} + \mathbf{F}\mathbf{u} - \mathbf{g} \in \mathbf{K} \text{ for some } \mathbf{u}\}, \quad (13)$$

where the cone \mathbf{K} is regular, i.e., it is closed, convex, pointed, and has a nonempty interior. Hence, the polar cone

$$\mathbf{K}^* = \{\mathbf{y} \mid \mathbf{y}'\mathbf{s} \geq 0 \ \forall \mathbf{s} \in \mathbf{K}\}$$

is also a regular cone (see the convex analysis in Rockafellar 1970). For technical reasons, we also assume that \mathcal{U} is a compact set with nonempty interior.

THEOREM 5. *The risk constraint $\eta_{\mathcal{U}}(v_0 + \mathbf{v}'\tilde{\mathbf{z}}) \leq 0$ is concisely representable as the conic constraints*

$$\begin{cases} v_0 + \mathbf{y}'\mathbf{g} \geq 0, \\ \mathbf{D}'\mathbf{y} = \mathbf{v}, \\ \mathbf{F}'\mathbf{y} = \mathbf{0}, \\ \mathbf{y} \in \mathbf{K}^*. \end{cases}$$

PROOF. The application of duality theory in formulating robust counterparts is well known (see, for example, Ben-Tal and Nemirovski 2001). Under the assumptions, the set \mathcal{U} satisfies the necessary Slater condition for strong duality. Therefore,

$$\eta_{\mathcal{U}}(v_0 + \mathbf{v}'\tilde{\mathbf{z}}) = -\min_{\mathbf{z}, \mathbf{u}} v_0 + \mathbf{v}'\mathbf{z} \quad \text{s.t. } \mathbf{D}\mathbf{z} + \mathbf{F}\mathbf{u} - \mathbf{g} \in \mathbf{K},$$

or equivalently,

$$\begin{aligned} \eta_{\mathcal{U}}(v_0 + \mathbf{v}'\tilde{\mathbf{z}}) &= -\max_{\mathbf{y}} v_0 + \mathbf{y}'\mathbf{g} \\ \text{s.t. } \mathbf{D}'\mathbf{y} &= \mathbf{v}, \\ \mathbf{F}'\mathbf{y} &= \mathbf{0}, \\ \mathbf{y} &\in \mathbf{K}^*. \end{aligned}$$

This results in the conic constraint representation of the feasible region. \square

5.2. Probability Bounds

In robust optimization, the conservativeness of the approach (equivalently, the tolerance to risk) is captured by the “size” of the uncertainty set. For example, one can think of \mathcal{U}_α as an uncertainty set of “size” α , where α is selected so that the probability of violating the constraint is not more than a prespecified constant $\epsilon(\alpha)$. More specifically, in view of the equivalence of optimization problems with chance constraints and the VaR formulation, α is selected so that

the corresponding robust risk measure is a conservative approximation of the $(1 - \epsilon)$ -VaR measure as follows:

$$\eta_{\mathcal{U}_\alpha}(v_0 + \mathbf{v}'\tilde{\mathbf{z}}) \geq \text{VaR}_{1-\epsilon(\alpha)}(v_0 + \mathbf{v}'\tilde{\mathbf{z}}) \quad \forall (v_0, \mathbf{v}) \in \mathfrak{R}^{m+1}. \quad (14)$$

Here $\epsilon(\alpha)$ typically decreases as α increases. The concern is whether the following remains true:

$$\eta_{\mathcal{U}_\alpha \cap \bar{\Omega}}(v_0 + \mathbf{v}'\tilde{\mathbf{z}}) \geq \text{VaR}_{1-\epsilon(\alpha)}(v_0 + \mathbf{v}'\tilde{\mathbf{z}}) \quad \forall (v_0, \mathbf{v}) \in \mathfrak{R}^{m+1}.$$

If it does, then making a risk measure coherent by using Theorem 4 does not increase the probability of constraint violation or, equivalently, it does not require a trade-off for coherent approximation of the $(1 - \epsilon)$ -VaR.

More generally, suppose a robust counterpart risk measure $\eta_{\mathcal{U}_\alpha}(v_0 + \mathbf{v}'\tilde{\mathbf{z}})$ is an upper bound of a risk measure $\rho(v_0 + \mathbf{v}'\tilde{\mathbf{z}})$ for all (v_0, \mathbf{v}) . We would like to know whether $\eta_{\mathcal{U}_\alpha \cap \bar{\Omega}}(v_0 + \mathbf{v}'\tilde{\mathbf{z}})$ remains an upper bound for $\rho(v_0 + \mathbf{v}'\tilde{\mathbf{z}})$. For this purpose, we assume that the set $\bar{\Omega}$ is compact with nonempty interior. We define the cone

$$\Pi = \text{cl}\{(\mathbf{z}, t) : \mathbf{z}/t \in \bar{\Omega}, t > 0\}.$$

Therefore, the cone Π and its dual cone Π^* are regular cones. Again, for technical reasons, we assume that the Slater condition for $\mathcal{U}_\alpha \cap \bar{\Omega}$ is satisfied.

THEOREM 6. *Let $\rho(\cdot)$ be a risk measure that satisfies the translation invariance and the monotonicity axioms. Suppose*

$$\eta_{\mathcal{U}_\alpha}(v_0 + \mathbf{v}'\tilde{\mathbf{z}}) \geq \rho(v_0 + \mathbf{v}'\tilde{\mathbf{z}}) \quad \forall (v_0, \mathbf{v}) \in \mathfrak{R}^{m+1}.$$

Then,

$$\eta_{\mathcal{U}_\alpha \cap \bar{\Omega}}(v_0 + \mathbf{v}'\tilde{\mathbf{z}}) \geq \rho(v_0 + \mathbf{v}'\tilde{\mathbf{z}}) \quad \forall (v_0, \mathbf{v}) \in \mathfrak{R}^{m+1}$$

if $\bar{\Omega} = \mathcal{C}\mathcal{H}(\Omega)$.

PROOF. Consider the following optimization problem:

$$\begin{aligned} -\eta_{\mathcal{U}_\alpha \cap \bar{\Omega}}(v_0 + \mathbf{v}'\tilde{\mathbf{z}}) &= \min_{\mathbf{z}} v_0 + \mathbf{v}'\mathbf{z} \\ \text{s.t. } \mathbf{z} &\in \mathcal{U}_\alpha, \\ (\mathbf{z}, 1) &\in \Pi, \end{aligned}$$

which is well defined in the compact set, and satisfies the Slater condition. Hence, the objective is the same as

$$\begin{aligned} \max_{(\mathbf{p}, t) \in \Pi^*} \left\{ \min_{\mathbf{z}: \mathbf{z} \in \mathcal{U}_\alpha} v_0 + \mathbf{v}'\mathbf{z} - \mathbf{z}'\mathbf{p} - t \right\} \\ = \min_{\mathbf{z}: \mathbf{z} \in \mathcal{U}_\alpha} v_0 + (\mathbf{v} - \mathbf{p}^*)'\mathbf{z} - t^* = -\eta_{\mathcal{U}_\alpha}(v_0 + (\mathbf{v} - \mathbf{p}^*)'\tilde{\mathbf{z}}) - t^* \end{aligned}$$

for some $(\mathbf{p}^*, t^*) \in \Pi^*$. Therefore,

$$\begin{aligned} \eta_{\mathcal{U}_\alpha \cap \bar{\Omega}}(v_0 + \mathbf{v}'\tilde{\mathbf{z}}) &= \eta_{\mathcal{U}_\alpha}(v_0 + (\mathbf{v} - \mathbf{p}^*)'\tilde{\mathbf{z}}) + t^* \\ &\geq \rho(v_0 + (\mathbf{v} - \mathbf{p}^*)'\tilde{\mathbf{z}}) + t^* \\ &\quad \text{because } \rho(v_0 + \mathbf{v}'\tilde{\mathbf{z}}) \leq \eta_{\mathcal{U}_\alpha}(v_0 + \mathbf{v}'\tilde{\mathbf{z}}) \\ &\quad \text{for all } (v_0, \mathbf{v}) \in \mathfrak{R}^{m+1} \\ &= \rho(v_0 + (\mathbf{v} - \mathbf{p}^*)'\tilde{\mathbf{z}} - t^*) \\ &\quad \text{(translation invariance).} \end{aligned}$$

Observe that $(\mathbf{z}, 1) \in \Pi$. Therefore, $\mathbf{p}^* \mathbf{z} + t^* \geq 0$. Hence, $v_0 + (\mathbf{v} - \mathbf{p}^*) \mathbf{z} - t^* \leq v_0 + \mathbf{v}' \mathbf{z}$, and by the monotonicity axiom,

$$\eta_{u_\alpha \cap \bar{\Omega}}(v_0 + \mathbf{v}' \bar{\mathbf{z}}) \geq \rho(v_0 + (\mathbf{v} - \mathbf{p}^*) \bar{\mathbf{z}} - t^*) \geq \rho(v_0 + \mathbf{v}' \bar{\mathbf{z}}). \quad \square$$

Note that the VaR measure satisfies the axioms of translation invariance and monotonicity. Therefore, optimization of a risk measure made coherent by using Theorem 4 is generally a more conservative approach in terms of the probability of constraint violation than optimization of the original risk measure.

5.3. A Numerical Example: Worst-Case VaR

In §4.1, we showed that the ellipsoidal uncertainty set maps to the mean-standard deviation risk measure. El Ghaoui et al. (2003) use this result to derive a formulation for the worst-case VaR based on first- and second-moment information about the distributions of uncertainties. However, formulating the problem using the ellipsoidal uncertainty set \mathcal{E}_α results in a noncoherent risk measure for general $\alpha > 0$.

We now provide a specific example of how one could make the resulting risk measure coherent. Suppose we have the additional information that

$$\bar{\Omega} = \{\mathbf{z}: -\underline{\mathbf{z}} \leq \mathbf{z} \leq \bar{\mathbf{z}}\} \subseteq \mathcal{CH}(\Omega).$$

Then, we can construct a coherent risk measure by intersecting the ellipsoidal uncertainty set with the set $\bar{\Omega}$. The robust counterpart of

$$\mathbf{x}' \mathbf{r} + \mathbf{x}' \mathbf{A} \mathbf{z} \geq 0 \quad \forall \mathbf{z} \in \mathcal{E}_\alpha \cap \bar{\Omega}$$

then reduces to the set of constraints

$$\begin{aligned} \mathbf{x}' \mathbf{r} &\geq \alpha \|\mathbf{A}' \mathbf{x} + \mathbf{t} - \mathbf{s}\|_2 + \bar{\mathbf{z}}' \mathbf{r} + \underline{\mathbf{z}}' \mathbf{s}, \\ \mathbf{t}, \mathbf{s} &\geq \mathbf{0}. \end{aligned} \tag{15}$$

El Ghaoui et al. (2003) discuss including support information in the worst-case VaR formulation, but do not relate it to the idea of coherence, and do not study the effect of the modified formulation on portfolio performance in computational experiments.

We explore the performance of (15) with a set of controlled numerical experiments. Consider a portfolio of $N = 20$ assets with uncertain returns $\tilde{r}_j, j = 1, \dots, N$. Each return \tilde{r}_j is determined by a simple single-factor model $\tilde{r}_j = r + \tilde{z}_j$. The factors \tilde{z}_j are independent and distributed as follows:

$$\tilde{z}_j = \begin{cases} \frac{\sqrt{\beta_j(1-\beta_j)}}{\beta_j} & \text{with probability } \beta_j, \\ -\frac{\sqrt{\beta_j(1-\beta_j)}}{(1-\beta_j)} & \text{with probability } 1-\beta_j. \end{cases}$$

Note that all random stock returns \tilde{r}_j have the same mean and standard deviation. However, depending on the parameters $\beta_j, j = 1, \dots, N$, the degree of symmetry of each individual return distribution can be different. Higher values for β_j (e.g., $\beta_j = 0.9$) result in large losses and small upside gains.

We conduct two sets of experiments. In both, we assume that $\mathbf{r} = \mathbf{e}$. In the first set (Return Distributions I), we generate values for β_j as follows:

$$\beta_j = \frac{1}{2} \left(1 + \frac{j}{N+1} \right), \quad j = 1, \dots, N.$$

All 20 return distributions are thus negatively skewed, and the return distributions for assets with high index numbers in the portfolio are more negatively skewed than those for stocks with low index numbers (the distribution for the first asset return is almost symmetric).

In the second set of experiments (Return Distributions II), we generate values for β_j as follows:

$$\beta_j = \frac{1}{8} \left(1 + \frac{j}{N+1} \right), \quad j = 1, \dots, N.$$

All 20 return distributions are thus positively skewed, and the return distributions for assets with high index numbers in the portfolio are less positively skewed than those for stocks with low index numbers (the distribution for the last asset return is almost symmetric).

The explicit formulations for the worst-case and the coherent worst-case VaR optimization problems are

(Worst-Case VaR)

$$\begin{aligned} \min \quad & \gamma \\ \text{s.t.} \quad & -\mathbf{x}' \mathbf{r} + \alpha \sqrt{\mathbf{x}' \Sigma \mathbf{x}} \leq \gamma, \\ & \mathbf{x}' \mathbf{r} \geq r_{\text{target}}, \\ & \mathbf{x}' \mathbf{e} = 1, \end{aligned} \tag{16}$$

and

(Coherent Worst-Case VaR)

$$\begin{aligned} \min \quad & \gamma \\ \text{s.t.} \quad & -\mathbf{x}' \mathbf{r} + \alpha \|\Sigma^{1/2} \mathbf{x} + \mathbf{t} - \mathbf{s}\|_2 + \bar{\mathbf{z}}' \mathbf{t} + \underline{\mathbf{z}}' \mathbf{s} \leq \gamma, \\ & \mathbf{x}' \mathbf{r} \geq r_{\text{target}}, \\ & \mathbf{x}' \mathbf{e} = 1, \\ & \mathbf{t}, \mathbf{s} \geq \mathbf{0}. \end{aligned} \tag{17}$$

We solve the optimization problems for different values of ϵ using the values of the distribution parameters in Return Distributions I (negative skew) and Return Distributions II (positive skew). The objective function values for the two optimization problems (i.e., the optimal VaRs expressed as returns) are presented in Table 1. The optimal VaR obtained by solving (16) (listed as “WVaR Obj”

Table 1. Optimal objective function values and realized VaRs (expressed as returns) obtained with the optimal weights from (16) and (17).

ϵ (%)	Return Distributions I (negative skew)					Return Distributions II (positive skew)				
	WVaR Obj	CWVaR Obj	WVaR Real	CWVaR Real	ESVaR	WVaR Obj	CWVaR Obj	WVaR Real	CWVaR Real	ESVaR
10.0	-0.329	-0.329	-0.702	-0.702	-0.521	-0.329	-0.612	-0.763	-0.612	-0.668
5.0	-0.025	-0.025	-0.612	-0.612	-0.501	-0.025	-0.612	-0.649	-0.612	-0.688
4.0	0.095	0.049	-0.571	0.049	-0.478	0.095	-0.612	-0.645	-0.612	-0.690
3.0	0.271	0.049	-0.543	0.049	-0.495	0.271	-0.612	-0.642	-0.612	-0.664
2.0	0.565	0.049	-0.494	0.049	-0.482	0.565	-0.612	-0.639	-0.612	-0.676
1.0	1.225	0.049	-0.430	0.049	-0.518	1.225	-0.612	-0.520	-0.612	-0.612
0.1	6.068	0.049	-0.281	0.049	-0.456	6.068	-0.612	-0.520	-0.612	-0.612

Note. The target expected return is assumed to be 1 in all optimization problems. For comparison, the value of the optimal sample VaR (ESVaR) is provided as well.

in Table 1) is naturally greater than or equal to the optimal VaR obtained by solving (17) (listed as “CWVaR Obj”) because optimization problem (17) is more constrained than (16). In particular, for small values of ϵ , the ellipsoid in the uncertainty set for returns becomes larger than the “box” set defined by the supports, and the optimal VaR from (16) is higher (worse) than the optimal VaR from (17).

We simulate a set of 1,000 realizations for returns, and estimate the realized (out-of-sample) VaRs, i.e., the actual sample VaRs for portfolios with the optimal portfolio weights from (16) and (17) (listed in columns “WVaR Real” and “CWVaR Real” in Table 1, respectively). For comparison purposes, we also compute the optimal exact sample VaR (“ESVaR”), which can be obtained as the optimal solution of the following mixed-integer (MIP) problem:

$$\begin{aligned}
 \min \quad & \gamma \\
 \text{s.t.} \quad & \gamma + (\mathbf{r}^i)' \mathbf{x} \geq -Ky_i, \quad i = 1, \dots, T, \\
 & \mathbf{x}' \mathbf{e} = 1, \\
 & \mathbf{y}' \mathbf{e} = \lfloor \epsilon T \rfloor, \\
 & \mathbf{y} \in \{0, 1\}^T,
 \end{aligned}$$

for some large constant K and a sample of T vectors of realized asset returns. Note that the optimal sample VaR problem is quite intractable, so we set a time limit of 1,800 seconds to the solver (CPLEX) we use to solve the problem. The solution we obtain, albeit not guaranteed to be optimal, is still useful for comparison purposes.

One can observe that the realized out-of-sample VaRs in Table 1 are always lower than the objective function value in the optimization problems, i.e., a portfolio manager can be confident that the VaR estimate she gets from solving the optimization problem would be conservative. We note that the VaR measures the maximum portfolio loss that may happen with probability ϵ , so it is desirable to have low numbers for the VaR value. The realized VaR performance differs depending on whether the distributions are positively or negatively skewed. In the case of positively

skewed asset returns, considering the intersection of the original ellipsoidal uncertainty set with the “box” set of supports improves the realized VaR.

Although the realized VaRs of optimal portfolios obtained by solving the coherent formulation (17) may or may not be lower than the realized VaRs of optimal portfolios obtained by solving the noncoherent formulation (16) depending on the characteristics of the asset return distributions, the performance of the optimal portfolios obtained by solving (17) is consistently better when it comes to maximum portfolio losses. Table 2 contains the realized maximum portfolio losses for the three portfolio VaR optimization formulations ((16), (17), and the exact sample VaR). One could therefore argue that including information about the support in the portfolio risk-minimization problem provides better worst-case performance. In this sense, making noncoherent portfolio risk measures coherent by incorporating support information imitates one of the effects of using the coherent risk measure CVaR in portfolio risk

Table 2. Maximum out-of-sample realized losses (expressed as returns) for the portfolios obtained by optimization of (16), (17), and exact sample VaR.

ϵ (%)	Return Distributions I (negative skew)			Return Distributions II (positive skew)		
	WVaR max	CWVaR max	ESVaR max	WVaR max	CWVaR max	ESVaR max
10.0	1.202	1.202	1.076	-0.329	-0.612	-0.763
5.0	1.202	1.202	1.076	-0.520	-0.612	-0.525
4.0	1.202	0.049	0.882	-0.520	-0.612	-0.527
3.0	1.202	0.049	1.340	-0.520	-0.612	-0.520
2.0	1.202	0.049	1.014	-0.520	-0.612	-0.612
1.0	1.202	0.049	1.141	-0.520	-0.612	-0.612
0.1	1.202	0.049	1.121	-0.520	-0.612	-0.612

Note. The target expected return is assumed to be 1 in all optimization problems.

minimization—namely, it shortens the tail of the distribution of portfolio losses.

6. Concluding Remarks

We present a unified view of risk measures in finance and uncertainty sets in robust optimization, and describe how robust optimization can be used to enhance the concepts of some risk measures. We also propose a practical approach to making existing risk measures coherent, and prove that the probability of constraint violation remains the same. Our computational experiments suggest that there may be practical benefits to using modified coherent risk measures with support information.

7. Electronic Companion

An electronic companion to this paper is available as part of the online version that can be found at <http://or.journal.informs.org/>.

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