Operations Research
doi 10.1287/opre.1080.0683ec

e-companion

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Electronic Companion—"Constructing Risk Measures from Uncertainty Sets" by Karthik Natarajan, Dessislava Pachamanova, and Melvyn Sim, *Operations Research*, doi 10.1287/opre.1080.0683.

Online Appendix

Proof of Theorem 4: Since $\rho(\cdot)$ is a proper coherent risk measure, the function $\rho(\boldsymbol{v}'\tilde{\boldsymbol{z}})$ is convex and positive homogenous in $\boldsymbol{v}$. Noting that the function $\rho(\boldsymbol{v}'\tilde{\boldsymbol{z}})$ is finite if the vector $\boldsymbol{v}$ is finite, the set $\Upsilon = \{(\boldsymbol{v}, u) : u \geq \rho(\boldsymbol{v}'\tilde{\boldsymbol{z}})\}$ is a closed convex cone as it is the epigraph of a convex positive homogeneous function. Moreover, the cone Υ is full-dimensional as we can find $n+2$ affinely independent points,

$$\left\{(\boldsymbol{0}, 0), (\boldsymbol{0}, 1), (\boldsymbol{e_1}, \rho(\boldsymbol{e_1}'\tilde{\boldsymbol{z}})), \ldots, (\boldsymbol{e_n}, \rho(\boldsymbol{e_n}'\tilde{\boldsymbol{z}}))\right\},$$

in the cone, where $\boldsymbol{e_i}$ denotes a unit vector with one at the ith element and zeros otherwise.
We next show that the cone Υ is also pointed. Suppose $(\boldsymbol{v}, t) \in \Upsilon$ and $(-\boldsymbol{v}\boldsymbol{y}, -t) \in \Upsilon$, that is

$$\rho(\boldsymbol{v}'\tilde{\boldsymbol{z}}) \leq t$$
$$\rho(-\boldsymbol{v}'\boldsymbol{z}) \leq -t.$$

Then we have, by subadditivity,

$$0 = \rho(0) \leq \rho(\boldsymbol{v}'\tilde{\boldsymbol{z}}) + \rho(-\boldsymbol{v}'\tilde{\boldsymbol{z}}) \leq t - t = 0. \tag{1}$$

However,

$$\rho(\boldsymbol{v}'\tilde{\boldsymbol{z}}) \geq \mathrm{E}(-\boldsymbol{v}'\tilde{\boldsymbol{z}}) = 0,$$

in which equality occurs when the variance of $\boldsymbol{y}'\tilde{\boldsymbol{z}}$ is zero. Since $\tilde{\boldsymbol{z}}$ has strictly positive definite covariance matrix, this condition holds only if $\boldsymbol{v} = \boldsymbol{0}$. Likewise

$$\rho(-\boldsymbol{v}'\tilde{\boldsymbol{z}}) \geq \mathrm{E}(\boldsymbol{v}'\tilde{\boldsymbol{z}}) = 0.$$

Hence, the equality (1) holds only if

$$\rho(\boldsymbol{v}'\tilde{\boldsymbol{z}}) = \rho(-\boldsymbol{v}'\tilde{\boldsymbol{z}}) = 0,$$

and that $\boldsymbol{v} = \boldsymbol{0}$. Likewise, since $\rho(\boldsymbol{v}'\tilde{\boldsymbol{z}}) \leq t \leq -\rho(-\boldsymbol{v}'\tilde{\boldsymbol{z}})$, we must have $t = 0$. Hence, Υ is a closed convex pointed cone with non-empty interior. Therefore, the dual cone, given by $\Upsilon^* = \{(\boldsymbol{z}, s) : (\boldsymbol{z}, s) \cdot (\boldsymbol{v}, t) \geq 0 \ \forall \ (\boldsymbol{v}, t) \in \Upsilon\}$ is also a closed convex cone with non-empty interior.
Observe that by translation invariance,

$$\rho\left(v_0 + \boldsymbol{v}'\tilde{\boldsymbol{z}}\right) = -v_0 + \rho\left(\boldsymbol{v}'\tilde{\boldsymbol{z}}\right) = -v_0 + \min_t\{t : (\boldsymbol{v}, t) \in \Upsilon\}$$

in which the optimization problem always has a finite optimal solution, $t^* = \rho(\boldsymbol{v}'\tilde{\boldsymbol{z}})$. Therefore, by strong conic duality, we have,

$$\begin{aligned}
\rho\left(v_0 + \boldsymbol{v}'\tilde{\boldsymbol{z}}\right) &= -v_0 + \min_t \{t : (\boldsymbol{v}, t) \in \Upsilon\} \\
&= -v_0 + \min_t \{t : (\boldsymbol{0}, 1)t - (-\boldsymbol{v}, 0) \in \Upsilon\} \\
&= -v_0 + \max_{\boldsymbol{z},s} \left\{-\boldsymbol{v}'\boldsymbol{z} : (\boldsymbol{z}, s) \in \Upsilon^*, s = 1\right\} \\
&= -\min_{\boldsymbol{z}} \left\{v_0 + \boldsymbol{v}'\boldsymbol{z} : (\boldsymbol{z}, 1) \in \Upsilon^*\right\}.
\end{aligned}$$

Hence

$$\rho\left(v_0 + \boldsymbol{v}'\tilde{\boldsymbol{z}}\right) = -\min_{\boldsymbol{z}\in\mathcal{C}} (v_0 + \boldsymbol{v}'\boldsymbol{z}),$$

with

$$\begin{aligned}
\mathcal{C} &= \{\boldsymbol{z} : (\boldsymbol{z}, 1) \in \Upsilon^*\} \\
&= \left\{\boldsymbol{z} \ : \ -\boldsymbol{v}'\boldsymbol{z} \leq s \ \forall \rho(\boldsymbol{v}'\tilde{\boldsymbol{z}}) \leq s\right\} \\
&= \left\{\boldsymbol{z} \ : \ -\boldsymbol{v}'\boldsymbol{z} \leq 1 \ \forall \rho(\boldsymbol{v}'\tilde{\boldsymbol{z}}) \leq 1\right\} \\
&= \left\{\boldsymbol{z} \ : \ \max_{\boldsymbol{y}} \{-\boldsymbol{y}'\boldsymbol{z} : \rho(\boldsymbol{y}'\tilde{\boldsymbol{z}}) \leq 1\} \leq 1\right\},
\end{aligned}$$

where the third inequality is due to the positive homogeneity of $\rho(\cdot)$ and $s \geq \rho(\boldsymbol{v}'\tilde{\boldsymbol{z}}) \geq 0$. Clearly, $\boldsymbol{0} \in \mathcal{C}$. To show that $\boldsymbol{0}$ is in the interior of $\mathcal{C}$, it suffices to show that for all $\boldsymbol{y} \neq \boldsymbol{0}$, $\max_{\boldsymbol{z}\in\mathcal{C}}(\boldsymbol{y}'\boldsymbol{z}) > 0$. Indeed, for all $\boldsymbol{y} \neq \boldsymbol{0}$, the variance of $\boldsymbol{y}'\tilde{\boldsymbol{z}}$ is strictly positive. Hence,

$$\max_{\boldsymbol{z}\in\mathcal{C}} (\boldsymbol{y}'\boldsymbol{z}) = -\min_{\boldsymbol{z}\in\mathcal{C}} (-\boldsymbol{y}'\boldsymbol{z}) = \rho(-\boldsymbol{y}'\tilde{\boldsymbol{z}}) > \mathrm{E}(\boldsymbol{y}'\tilde{\boldsymbol{z}}) = 0.$$

To prove that $\mathcal{C} \subseteq \mathcal{CH}(\Omega)$, it suffices to show that for all $\boldsymbol{y}$

$$\min_{\boldsymbol{z}\in\mathcal{C}} \boldsymbol{y}'\boldsymbol{z} \geq \min_{\boldsymbol{z}\in\mathcal{CH}(\Omega)} \boldsymbol{y}'\boldsymbol{z}.$$

Indeed,

$$\begin{aligned}
-\min_{\boldsymbol{z}\in\mathcal{C}} \boldsymbol{y}'\boldsymbol{z} &= \rho(\boldsymbol{y}'\tilde{\boldsymbol{z}}) \\
&\leq \rho\left(\min_{\boldsymbol{z}\in\mathcal{CH}(\Omega)} \boldsymbol{y}'\boldsymbol{z}\right) \\
&= -\min_{\boldsymbol{z}\in\mathcal{CH}(\Omega)} \boldsymbol{y}'\boldsymbol{z},
\end{aligned}$$

where the inequality is due to the Axiom of Monotonicity and noting that $\min_{\boldsymbol{z}\in\mathcal{CH}(\Omega)} \boldsymbol{y}'\boldsymbol{z} \leq \boldsymbol{y}'\tilde{\boldsymbol{z}}$.

We now prove the converse, that is, for any convex uncertainty set with $\boldsymbol{0}$ in the interior and $\mathcal{C} \subseteq \mathcal{CH}(\Omega)$, the robust counterpart risk measure $\eta_{\mathcal{C}}(v_0 + \boldsymbol{v}'\tilde{\boldsymbol{z}})$ is a proper coherent risk measure defined $\mathcal{V}$. We first show that the risk measure is proper, which is equivalent to saying that for all $\boldsymbol{v} \neq \boldsymbol{0}$,

$$\eta_{\mathcal{C}}(v_0 + \boldsymbol{v}'\tilde{\boldsymbol{z}}) > \mathrm{E}(-v_0 - \boldsymbol{v}'\tilde{\boldsymbol{z}}) = -v_0.$$

Since $\mathbf{0}$ is in the interior of $\mathcal{C}$ we have for all $\boldsymbol{v} \neq \mathbf{0}$,

$$\eta_{\mathcal{C}}(v_0 + \boldsymbol{v}'\tilde{\boldsymbol{z}}) = -v_0 - \underbrace{\min_{\boldsymbol{z}\in\mathcal{C}}(\boldsymbol{v}'\boldsymbol{z})}_{<0} > -v_0.$$

It is trivial to show translation invariance and positive homogeneity. With regard to subadditivity, we consider two random variables in $\mathcal{V}$, $v_0 + \boldsymbol{v}'\tilde{\boldsymbol{z}}$ and $w_0 + \boldsymbol{w}'\tilde{\boldsymbol{z}}$. Observe that

$$\begin{aligned}
\eta_{\mathcal{C}}(v_0 + \boldsymbol{v}'\tilde{\boldsymbol{z}} + w_0 + \boldsymbol{w}'\tilde{\boldsymbol{z}}) &= -\min_{\boldsymbol{z}\in\mathcal{C}}(v_0 + w_0 + (\boldsymbol{v}+\boldsymbol{w})'\boldsymbol{z}) \\
&\leq \left(-\min_{\boldsymbol{z}\in\mathcal{C}} v_0 + \boldsymbol{v}'\boldsymbol{z}\right) + \left(-\min_{\boldsymbol{z}\in\mathcal{C}} w_0 + \boldsymbol{w}'\boldsymbol{z}\right) \\
&= \eta_{\mathcal{C}}(v_0 + \boldsymbol{v}'\tilde{\boldsymbol{z}}) + \eta_{\mathcal{C}}(w_0 + \boldsymbol{w}'\tilde{\boldsymbol{z}}).
\end{aligned}$$

To show monotonicity, we note that if $v_0 + \boldsymbol{v}'\tilde{\boldsymbol{z}} \geq 0$, then

$$\min_{\boldsymbol{z}\in\mathcal{CH}(\Omega)} v_0 + \boldsymbol{v}'\boldsymbol{z} \geq 0.$$

For $\mathcal{C} \subseteq \mathcal{CH}(\Omega)$, we have

$$\eta_{\mathcal{C}}(v_0 + \boldsymbol{v}'\tilde{\boldsymbol{z}}) = -\min_{\boldsymbol{z}\in\mathcal{C}} v_0 + \boldsymbol{v}'\boldsymbol{z} \leq -\min_{\boldsymbol{z}\in\mathcal{CH}(\Omega)} v_0 + \boldsymbol{v}'\boldsymbol{z} \leq 0.$$

Suppose $v_0 + \boldsymbol{v}'\tilde{\boldsymbol{z}} \geq w_0 + \boldsymbol{w}'\tilde{\boldsymbol{z}}$, or equivalently, $v_0 - w_0 + (\boldsymbol{v} - \boldsymbol{w})'\tilde{\boldsymbol{z}} \geq 0$, we have

$$\begin{aligned}
\eta_{\mathcal{C}}(v_0 + \boldsymbol{v}'\tilde{\boldsymbol{z}}) &\leq \eta_{\mathcal{C}}(v_0 + \boldsymbol{v}'\tilde{\boldsymbol{z}} - (w_0 + \boldsymbol{w}'\tilde{\boldsymbol{z}})) + \eta_{\mathcal{C}}(w_0 + \boldsymbol{w}'\tilde{\boldsymbol{z}}) \quad \text{(subadditivity)} \\
&= \underbrace{\eta_{\mathcal{C}}(v_0 - w_0 + (\boldsymbol{v} - \boldsymbol{w})'\tilde{\boldsymbol{z}})}_{\leq 0} + \eta_{\mathcal{C}}(w_0 + \boldsymbol{w}'\tilde{\boldsymbol{z}}) \\
&\leq \eta_{\mathcal{C}}(w_0 + \boldsymbol{w}'\tilde{\boldsymbol{z}}),
\end{aligned}$$

which yields the desired result. ■