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ONLINE SUPPORTING MATERIALS FOR MIXED ZERO-ONE LINEAR PROGRAMS UNDER OBJECTIVE UNCERTAINTY: A COMPLETELY POSITIVE REPRESENTATION

Appendix I. Formulations of Related Moment Models

Marginal Moment Model (MMM).

Consider the special case of $Z(\tilde{\mathbf{c}})$ with $\mathcal{B} = \{1, \dots, n\}$, and denote it as $Z_{01}(\tilde{\mathbf{c}})$:

$$
\begin{aligned}
Z_{01}(\tilde{\mathbf{c}}) = \quad & \max \quad \tilde{\mathbf{c}}^T \mathbf{x} \\
& \text{s.t.} \quad \mathbf{a}_i^T \mathbf{x} = b_i \qquad \forall i = 1, \dots, m \\
& \qquad\quad x_j \in \{0, 1\} \quad \forall j = 1, \dots, n
\end{aligned}
\tag{0.1}
$$

Bertsimas, Natarajan and Teo [6] solve the following problem:

$$\sup_{\tilde{c}_j \sim (\mu_j, \Sigma_{jj})^+, \forall j = 1, \dots, n} \mathbf{E}\left[Z_{01}(\tilde{\mathbf{c}})\right]$$

under the assumption that the convex hull of the 0-1 problem is given by the linear constraints

$$\left\{\mathbf{x} \middle| \, \mathbf{a}_i^T \mathbf{x} = b_i, \forall i = 1, \dots, m, \, 0 \le x_j \le 1, \forall j = 1, \dots, n \right\}.$$

The SDP formulation they developed for this problem is:

$$
\begin{aligned}
\text{(MMM)} \qquad & \sup \quad \sum_{j=1}^{n} y_j \\
& \text{s.t.} \quad \mathbf{a}_i^T \mathbf{P} = b_i \qquad \forall i = 1, \dots, m \\
& \qquad\quad \mu_j \ge y_j \ge 0 \qquad \forall j = 1, \dots, n \\
& \qquad\quad \begin{pmatrix} 1 & \mu_j \\ \mu_j & \Sigma_{jj} \end{pmatrix} \succeq \begin{pmatrix} p_j & y_j \\ y_j & z_j \end{pmatrix} \succeq 0 \quad \forall j = 1, \dots, n
\end{aligned}
$$

The variables in this formulation can be interpreted as

$$
\begin{aligned}
p_j &= \mathbf{P}(x_j(\tilde{\mathbf{c}}) = 1), \\
y_j &= \mathbf{E}[\tilde{c}_j | x_j(\tilde{\mathbf{c}}) = 1]\mathbf{P}(x_j(\tilde{\mathbf{c}}) = 1), \\
z_j &= \mathbf{E}[\tilde{c}_j^2 | x_j(\tilde{\mathbf{c}}) = 1]\mathbf{P}(x_j(\tilde{\mathbf{c}}) = 1).
\end{aligned}
$$

where $x_j(\mathbf{c})$ is the optimal value of x_j under objective $\mathbf{c}$. The value p_j in the optimal solution is the persistency of corresponding variable under the extremal distribution.

Cross Moment Model (CMM).

Consider the special case of $Z(\tilde{\mathbf{c}})$ with $\mathcal{B} = \emptyset$ and denote it as $Z_{LP}(\tilde{\mathbf{c}})$:

$$
\begin{aligned}
Z_{LP}(\tilde{\mathbf{c}}) = \quad & \max \quad \tilde{\mathbf{c}}^T \mathbf{x} \\
& \text{s.t.} \quad A\mathbf{x} = \mathbf{b} \\
& \qquad\quad \mathbf{x} \ge \mathbf{0}
\end{aligned}
\tag{0.2}
$$

Let $\mathcal{BASIS}$ index the set of all basic feasible solutions to this linear program and $\mathbf{x}^{(j)}$ be the jth basic feasible solution. Bertsimas, Vinh, Natarajan and Teo [4] solve the following problem:

$$\sup_{\tilde{\mathbf{c}}\sim(\boldsymbol{\mu},\Sigma)} \mathbf{E}\left[Z_{LP}(\tilde{\mathbf{c}})\right]$$

The SDP formulation they developed for this problem is:

$$\begin{array}{llll} \text{(CMM)} & \sup & \displaystyle\sum_{j\in\mathcal{BASIS}} \mathbf{x}^{(j)T}\mathbf{y}_j & \\ & \text{s.t.} & \displaystyle\sum_{j\in\mathcal{BASIS}} \begin{pmatrix} p_j & \mathbf{y}_j^T \\ \mathbf{y}_j & Z_j \end{pmatrix} = \begin{pmatrix} 1 & \boldsymbol{\mu}^T \\ \boldsymbol{\mu} & \Sigma \end{pmatrix} & \\ & & \begin{pmatrix} p_j & \mathbf{y}_j^T \\ \mathbf{y}_j & Z_j \end{pmatrix} \succeq 0 & \forall j \in \mathcal{BASIS} \end{array}$$

The variables $p_j \in \mathbb{R}$, $\mathbf{y}_j \in \mathbb{R}^n$ and $Z_j \in \mathcal{S}_n$ in this formulation can be interpreted as

$$\begin{aligned} p_j &= \mathbf{P}(\text{Basis } j \text{ is optimal}), \\ \mathbf{y}_j &= \mathbf{E}[\tilde{\mathbf{c}}|\text{Basis } j \text{ is optimal}], \\ Z_j &= \mathbf{E}[\tilde{\mathbf{c}}\tilde{\mathbf{c}}^T|\text{Basis } j \text{ is optimal}]. \end{aligned}$$

The exponential number of basic feasible solutions for linear programs makes this formulation very large and difficult to use for general linear programs.

Generalized Chebyshev Bounds.

Vandenberghe, Boyd and Comanor [40] consider a generalization of Chebyshev's inequality:

$$\inf_{\tilde{\mathbf{c}}\sim(\boldsymbol{\mu},\Sigma)} \mathbf{P}\left(\mathbf{c}^T A_i \mathbf{c} + 2\mathbf{b}_i^T\mathbf{c} + d_i < 0,\ \forall i = 1,\dots,m\right)$$

The SDP formulation they proposed for this problem is:

$$\begin{array}{lll} \min & 1 - \displaystyle\sum_{i=1}^m \lambda_i & \\ \text{s.t.} & \mathrm{tr}(A_i Z_i) + 2\mathbf{b}_i^T\mathbf{z}_i + d_i\lambda_i \geq 0 & \forall i = 1,\dots,m \\ & \displaystyle\sum_{i=1}^m \begin{pmatrix} \lambda_i & \mathbf{z}_i^T \\ \mathbf{z}_i & Z_i \end{pmatrix} \preceq \begin{pmatrix} 1 & \boldsymbol{\mu}^T \\ \boldsymbol{\mu} & \Sigma \end{pmatrix} & \\ & \begin{pmatrix} \lambda_i & \mathbf{z}_i^T \\ \mathbf{z}_i & Z_i \end{pmatrix} \succeq 0 & \forall i = 1,\dots,m \end{array}$$

where the variables are $Z_i \in \mathcal{S}_n$, $\mathbf{z}_i \in \mathbb{R}^n$, and $\lambda_i \in \mathbb{R}$, $\forall i = 1,\dots,m$. The VBC approach can be used in stochastic sensitivity analysis for linear programming problems. Let $\mathcal{B}$ be the index set of the basic variables in a basic feasible solution. The reduced cost $\bar{c}_j$ of the variable x_j is defined as

$$\bar{c}_j := c_j - \mathbf{c}_{\mathcal{B}}^T A_{\mathcal{B}}^{-1} A_j,$$

where $A_{\mathcal{B}}$ is the columns of A indexed by $\mathcal{B}$, and A_j is the jth column of A. Let $\mathcal{N} = \{1,\dots,n\}\setminus\mathcal{B}$ be the index set of the nonbasic variables. In order for a given feasible solution of the linear program to be optimal, the reduced cost of all the nonbasic variables must be nonpositive (for maximization problems). This defines a set of linear inequalities on $\tilde{\mathbf{c}}$ that we can exploit in the VBC approach, i.e. $\forall j \in \mathcal{N}$,

$$\begin{aligned}\bar{\tilde{c}}_j \leq 0 &\iff \tilde{c}_j - \tilde{\mathbf{c}}_{\mathcal{B}}^T A_{\mathcal{B}}^{-1} A_j \leq 0 \\ &\iff \mathbf{e}_j^T \tilde{\mathbf{c}}_{\mathcal{N}} - \left(A_{\mathcal{B}}^{-1} A_j\right)^T \tilde{\mathbf{c}}_{\mathcal{B}} \leq 0 \\ &\iff \begin{pmatrix} -A_{\mathcal{B}}^{-1} A_j \\ \mathbf{e}_j \end{pmatrix}^T \begin{pmatrix} \tilde{\mathbf{c}}_{\mathcal{B}} \\ \tilde{\mathbf{c}}_{\mathcal{N}} \end{pmatrix} \leq 0,\end{aligned}$$

where $\mathbf{e}_j$ denotes the unit vector with one on its j^{th} entry and zeros eleswhere.

Hence, these $|\mathcal{N}|$ inequalities can be viewed as a set constraining $\tilde{\mathbf{c}}$,

$$C_{LP} = \left\{ \mathbf{c} \in \mathbb{R}^n \;\middle|\; \begin{pmatrix} -A_{\mathcal{B}}^{-1} A_j \\ \mathbf{e}_j \end{pmatrix}^T \mathbf{c} \leq 0, \, \forall j \in \mathcal{N} \right\}.$$

Then any realization of $\tilde{\mathbf{c}}$ falling in this set (i.e. $\tilde{\mathbf{c}} \in C_{LP}$) will make the pre-given feasible solution optimal. Therefore, the probability that $\tilde{\mathbf{c}}$ lies in the set is just the probability that the given feasible solution is optimal. Furthermore, when A is of rank one, i.e. there is only one basic variable in any feasible solution, that probability is just the persistency of that particular basic variable. One difference between C_{LP} and C in the VBC approach is that all the inequalities in C are strict, so when we apply the VBC model to this problem, the interpretation of the resulting probability has to be changed to the probability that the given feasible solution is the unique optimum. Using the VBC approach on the LP problem, we obtain:

$$\text{(VBC)} \qquad \begin{array}{lll} \min & 1 - \sum\limits_{j \in \mathcal{N}} \lambda_j & \\ \text{s.t.} & \begin{pmatrix} -A_{\mathcal{B}}^{-1} A_j \\ \mathbf{e}_j \end{pmatrix}^T \mathbf{z}_j \geq 0 & \forall j \in \mathcal{N} \\ & \sum\limits_{j \in \mathcal{N}} \begin{pmatrix} Z_j & \mathbf{z}_j \\ \mathbf{z}_j^T & \lambda_j \end{pmatrix} \preceq \begin{pmatrix} \Sigma & \boldsymbol{\mu} \\ \boldsymbol{\mu}^T & 1 \end{pmatrix} & \\ & \begin{pmatrix} Z_j & \mathbf{z}_j \\ \mathbf{z}_j^T & \lambda_j \end{pmatrix} \succeq 0 & \forall j \in \mathcal{N} \end{array}$$

Thus, for any given feasible solution, we can solve the corresponding (VBC) and obtain the optimal objective value, which is the tightest lower bound on the probability that the given feasible solution is the unique optimal solution to Problem (0.2).

Appendix II. Copositive and Completely Positive Programs

The materials in this section are based on [3, 10, 15].

Properties.

The cone of completely positive matrices, positive semidefinite matrices, copositive matrices and symmetric matrices satisfy

$$\mathcal{CP}_n \subsetneq \mathcal{S}_n^+ \subsetneq \mathcal{CO}_n \subsetneq \mathcal{S}_n,$$

and all these cones are pointed and closed convex.

Proposition. *A is completely positive if and only if there exist vectors* $\mathbf{v}_1, \mathbf{v}_2 \ldots, \mathbf{v}_k \in \mathbb{R}_+^n$ *such that*

$$A = \sum_{i=1}^{k} \mathbf{v}_i \mathbf{v}_i^T. \tag{0.3}$$

Clearly, the factorization of a completely positive matrix is not unique. The representation (0.3) is called a *rank* 1 *representation* of A. The decomposition of A into the sum of rank 1 matrices is referred to as a *completely positive decomposition*. The minimal k for which there exists a rank 1 representation is called the *cp-rank* of A. From the definition, it is clear that $cp - rank(A) \geq rank(A)$ for every CP matrix A. Equality holds when $n \leq 3$, or when $rank(A) \leq 2$. An upper bound on cp-rank of A in terms of rank A (when $rankA \geq 2$) is

$$cp - rank(A) \leq \frac{rank(A)(rank(A)+1)}{2} - 1.$$

Duality.

For a pointed closed convex cone $\mathcal{K}$, its *dual cone* $\mathcal{K}^*$ is defined as:

$$\mathcal{K}^* := \left\{ A \in \mathcal{S}_n \,\middle|\, \forall B \in \mathcal{K},\, \mathrm{tr}(AB) \geq 0 \right\}.$$

$\mathcal{K}^*$ is also a pointed closed convex cone. The following shows a pair of primal-dual conic optimization problems:

$$\begin{array}{lll} \text{(Primal)} & \min & C \bullet X \\ & \text{s.t.} & A_i \bullet X = b_i \quad i = 1, \ldots, m \\ & & X \in \mathcal{K} \end{array} \tag{0.4}$$

$$\begin{array}{lll} \text{(Dual)} & \max & b^T y \\ & \text{s.t.} & S = C - \sum_{i=1}^{m} y_i A_i \\ & & S \in \mathcal{K}^* \end{array} \tag{0.5}$$

If both problems have a strictly feasible point, i.e. $\exists X \in int(\mathcal{K})$, such that

$$A_i \bullet X = b_i,\, \forall i = 1, \ldots, m,$$

and $\exists S \in int(\mathcal{K}^*)$, such that

$$S = C - \sum_{i=1}^{m} y_i A_i,$$

then Problem (0.4) and (0.5) are equivalent, i.e. the duality gap $C \bullet X^* - b^T y^*$ is 0 at optimality, where X^* and y^* are the optimal solutions to Problem (0.4) and (0.5) respectively. We verify next that the cones $\mathcal{CO}_n$ and $\mathcal{CP}_n$ are dual cones in $\mathcal{S}_n$.

Theorem. *$\mathcal{CP}_n^* = \mathcal{CO}_n$ and $\mathcal{CO}_n^* = \mathcal{CP}_n$.*

Proof. We first prove $\mathcal{CP}_n^* = \mathcal{CO}_n$, and then $\mathcal{CO}_n^* = \mathcal{CP}_n$ will follow since both cones are closed. Let $A \in \mathcal{S}_n$. Then

$$\begin{aligned} A \in \mathcal{CP}_n^* &\iff \forall B \in \mathcal{CP}_n,\ \mathrm{tr}(AB) \geq 0 \\ &\iff \forall V \in \mathbb{R}_+^{n\times k},\ \mathrm{tr}(AVV^T) \geq 0 \\ &\iff \forall V \in \mathbb{R}_+^{n\times k},\ \mathrm{tr}(V^T A V) \geq 0 \\ &\iff \forall \mathbf{v} \in \mathbb{R}_+^{n},\ \mathbf{v}^T A \mathbf{v} \geq 0 \\ &\iff A \in \mathcal{CO}_n. \end{aligned}$$

Thus, $\mathcal{CP}_n^* = \mathcal{CO}_n$. □

Similarly, it can be shown that $\mathcal{S}_n^{+*} = \mathcal{S}_n^+$, i.e. $\mathcal{S}_n^+$ is self-dual. The interior of the completely positive and copositive cone is characterized as:

$$int(\mathcal{CP}_n) = \left\{ A \in \mathcal{S}_n \,\middle|\, \exists V_1 > 0 \text{ nonsingular },\ V_2 \geq 0, \text{ such that } A = [V_1|V_2][V_1|V_2]^T \right\}.$$

$$int(\mathcal{CO}_n) = \left\{ A \in \mathcal{S}_n \,\middle|\, \forall \mathbf{v} \in \mathbb{R}_+^n, \mathbf{v} \neq \mathbf{0},\ \mathbf{v}^T A \mathbf{v} > 0 \right\}.$$

The notation $[V_1|V_2]$ describes the matrix whose columns are the columns of V_1 augmented with the columns of V_2.

Approximating the Copositive Cone and the Completely Positive Cone.

Klerk and Pasechnik [21] show that there exists a series of linear and semidefinite representable cones approximating the copositive cone $\mathcal{CO}_n$ from the inside, i.e.

$$\begin{gathered} \exists \text{ closed convex cones } \{\mathcal{K}_n^r : r = 0, 1, 2, \dots\} \\ \text{such that } \mathcal{K}_n^r \subseteq \mathcal{K}_n^{r+1},\ \forall r \geq 0 \text{ and } \overline{\bigcup_{r\geq 0} \mathcal{K}_n^r} = \mathcal{CO}_n. \end{gathered}$$

The dual cones $\{(\mathcal{K}_n^r)^* : r = 0, 1, 2, \dots\}$ approximate the completely positive cones $\mathcal{CP}_n$ from outside, i.e.

$$(\mathcal{K}_n^r)^* \supseteq \left(\mathcal{K}_n^{r+1}\right)^*,\ \forall r \geq 0 \text{ and } \bigcap_{r\geq 0} (\mathcal{K}_n^r)^* = \mathcal{CO}_n^* = \mathcal{CP}_n.$$

For example, when $r = 0$, we have

$$\begin{gathered} \mathcal{K}_n^0 = \left\{ A \in \mathcal{S}_n \,\middle|\, \exists X \in \mathcal{S}_n^+,\ \exists Y \in \mathbb{R}_+^{n\times n},\ A = X + Y \right\}, \\ (\mathcal{K}_n^0)^* = \left\{ A \in \mathcal{S}_n^+ \,\middle|\, A \in \mathbb{R}_+^{n\times n} \right\}. \end{gathered}$$

It can be shown that when $n \leq 4$, the above two approximations are exact, i.e. $\mathcal{K}_n^0 = \mathcal{CO}_n$ and $(\mathcal{K}_n^0)^* = \mathcal{CP}_n$.

The higher order approximation ($r \geq 1$) becomes much more complicated. For instance, when $r = 1$, Parrilo (2000, [34]) showed that

$$\mathcal{K}_n^1 = \left\{ A \in \mathcal{S}_n \;\middle|\; \begin{array}{r} \exists M^{(i)} \in \mathcal{S}_n,\, i = 1, 2 \ldots, n \\ \text{such that} \end{array} \left\{ \begin{array}{l} A - M^{(i)} \succeq 0,\, i = 1, 2 \ldots, n \\ M_{ii}^{(i)} = 0,\, i = 1, 2 \ldots, n \\ M_{jj}^{(i)} + M_{ij}^{(j)} + M_{ji}^{(j)} = 0,\, i \neq j \\ M_{jk}^{(i)} + M_{ik}^{(j)} + M_{ij}^{(k)} \geq 0,\, i \neq j \neq k \end{array} \right. \right\}.$$

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