Introduction

Expectations

Densities 3 and 4 steps

Probability Densities of Random Walks

James Wan

The University of Newcastle

9 July, 2010

Co-authors: Jon Borwein, Armin Straub, (Wadim Zudilin)
Outline

1. Introduction

2. Expectations
   - Experimental maths 1

3. Densities

4. 3 and 4 steps
   - Experimental maths 2
The random walk integrals

**Definition**

\[ W_n(s) := \int_{[0,1]^n} \left| \sum_{k=1}^{n} e^{2\pi x_k i} \right|^s \, d\mathbf{x} \]

for complex \( s \). \( W_n := W_n(1) \).

**Definition**

Let \( p_n \) be the (unique) function that satisfies

\[ W_n(s) = \int_0^n p_n(x)x^s \, dx. \]
The random walk integrals

Definition

\[ W_n(s) := \int_{[0,1]^n} \left| \sum_{k=1}^{n} e^{2\pi x_k i} \right|^s d\mathbf{x} \]

for complex \( s \). \( W_n := W_n(1) \).

Definition

Let \( p_n \) be the (unique) function that satisfies

\[ W_n(s) = \int_{0}^{n} p_n(x)x^s dx. \]

- Work in progress...
The random walk integrals

**Definition**

\[ W_n(s) := \int_{[0,1]^n} \left| \sum_{k=1}^{n} e^{2\pi x_k i} \right|^s \, d\mathbf{x} \]

for complex \( s \). \( W_n := W_n(1) \).

**Definition**

Let \( p_n \) be the (unique) function that satisfies

\[ W_n(s) = \int_{0}^{n} p_n(x) x^s \, dx. \]

- Work in progress...
- Makes heavy use of experimental mathematics.
  
  “Had one not erred, one would have achieved less.”
What we know

- $W_1(s) = 1$, $W_2(s) = \binom{s}{s/2}$. So $p_1(x) = \delta_1(x)$,
  
  $p_2(x) = \frac{2}{\pi \sqrt{4-x^2}}$. 

What we know

- $W_1(s) = 1$, $W_2(s) = \binom{s}{s/2}$. So $p_1(x) = \delta_1(x)$,
  \[ p_2(x) = \frac{2}{\pi \sqrt{4-x^2}}. \]

- $W_3(\pm 1)$ have closed form, rest follows by recursion.
What we know

- $W_1(s) = 1$, $W_2(s) = \binom{s}{s/2}$. So $p_1(x) = \delta_1(x)$,
  \[ p_2(x) = \frac{2}{\pi \sqrt{4-x^2}}. \]

- $W_3(\pm 1)$ have closed form, rest follows by recursion.

- Later: part of derivation for $W_4(\pm 1)$.
What we know

- \( W_1(s) = 1, \ W_2(s) = \binom{s}{s/2} \). So \( p_1(x) = \delta_1(x) \),
  \[ p_2(x) = \frac{2}{\pi \sqrt{4-x^2}}. \]

- \( W_3(\pm1) \) have closed form, rest follows by recursion.

- Later: part of derivation for \( W_4(\pm1) \).

- \( p_n \) is unique as all moments are known and the interval of integration is finite.
What we know

- $W_1(s) = 1$, $W_2(s) = \binom{s}{s/2}$. So $p_1(x) = \delta_1(x)$,
  $$p_2(x) = \frac{2}{\pi\sqrt{4-x^2}}.$$

- $W_3(\pm 1)$ have closed form, rest follows by recursion.

- Later: part of derivation for $W_4(\pm 1)$.

- $p_n$ is unique as all moments are known and the interval of integration is finite.

- We shift focus from $W_n$ to $p_n$, in particular $p_3$ and $p_4$. 
Outline

1. Introduction

2. Expectations
   - Experimental maths 1

3. Densities

4. 3 and 4 steps
   - Experimental maths 2
Closed forms


\[ W_4(-1) = \frac{\pi}{4} \, \frac{7}{5} F_6 \left( \begin{array}{c} \frac{1}{2} \, , \, \frac{1}{2} \, , \, \frac{1}{2} \, , \, \frac{1}{2} \, , \, \frac{1}{2} \, , \, \frac{1}{2} \\ \frac{1}{4} \, , \, 1 \, , \, 1 \, , \, 1 \, , \, 1 \, , \, 1 \end{array} \right| 1 \). \]
Closed forms


\[ W_4(-1) = \frac{\pi}{4} \, 7F_6 \left( \begin{array}{c} \frac{5}{4}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ \frac{1}{4}, 1, 1, 1, 1, 1 \end{array} \mid 1 \right). \]


Both of the following are equal to \( W_4(1) \):

\[
\frac{3\pi}{4} \, 7F_6 \left( \begin{array}{c} \frac{7}{4}, \frac{3}{2}, \frac{3}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ \frac{3}{4}, 2, 2, 2, 1, 1 \end{array} \mid 1 \right) - \frac{3\pi}{8} \, 7F_6 \left( \begin{array}{c} \frac{7}{4}, \frac{3}{2}, \frac{3}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ \frac{3}{4}, 2, 2, 2, 2, 1 \end{array} \mid 1 \right)
\]

\[
= \frac{9\pi}{4} \, 7F_6 \left( \begin{array}{c} \frac{7}{4}, \frac{3}{2}, \frac{3}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ \frac{3}{4}, 2, 2, 2, 1, 1 \end{array} \mid 1 \right) - 2\pi \, 7F_6 \left( \begin{array}{c} \frac{5}{4}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ \frac{1}{4}, 1, 1, 1, 1, 1 \end{array} \mid 1 \right).
\]
Proof of Theorem (1)

The proof uses *Bailey’s identity* connecting $G_{4,4}^{2,4}$ to $7F_6$. 

But recall that $W_4(-1)$ is a $G_{2,2}^{4,4}$. Fear not! For we use the definition of Meijer G-functions to obtain the integrand for $W_4(-1)$:

$$
\Gamma\left(\frac{1}{2} - t\right) \frac{\Gamma(t)}{\Gamma(1-t)} \cdot \sin^2\left(\frac{\pi t}{2}\right)
$$

using $\Gamma(t)\Gamma(1-t) = \frac{\pi}{\sin(\pi t)}$. We choose the contour to enclose the poles of $\Gamma\left(\frac{1}{2} - t\right)$.
The proof uses *Bailey’s identity* connecting $G_{4,4}^{2,4}$ to $7F_6$.

But recall that $W_4(-1)$ is a $G_{4,4}^{2,2}$. 
Proof of Theorem (1)

The proof uses Bailey’s identity connecting $G_{4,4}^{2,4}$ to $7F_6$.

But recall that $W_4(-1)$ is a $G_{4,4}^{2,2}$.

Fear not! For we use the definition of Meijer G-functions to obtain the integrand for $W_4(-1)$:

$$
\frac{\Gamma(1/2 - t)^2 \Gamma(t)^2}{\Gamma(1/2 + t)^2 \Gamma(1 - t)^2} x^t = \frac{\Gamma(1/2 - t)^2 \Gamma(t)^4}{\Gamma(1/2 + t)^2} \cdot \frac{\sin^2(\pi t)}{\pi^2} x^t,
$$

using $\Gamma(t)\Gamma(1 - t) = \pi / \sin(\pi t)$. 
Proof of Theorem (1)

The proof uses *Bailey’s identity* connecting $G_{4,4}^{2,4}$ to $7F_6$.

But recall that $W_4(-1)$ is a $G_{4,4}^{2,2}$.

Fear not! For we use the definition of Meijer G-functions to obtain the integrand for $W_4(-1)$:

$$
\frac{\Gamma\left(\frac{1}{2} - t\right)^2 \Gamma(t)^2}{\Gamma\left(\frac{1}{2} + t\right)^2 \Gamma(1 - t)^2} x^t = \frac{\Gamma\left(\frac{1}{2} - t\right)^2 \Gamma(t)^4}{\Gamma\left(\frac{1}{2} + t\right)^2} \cdot \frac{\sin^2(\pi t)}{\pi^2} x^t,
$$

using $\Gamma(t)\Gamma(1 - t) = \pi / \sin(\pi t)$.

We choose the contour to enclose the poles of $\Gamma\left(\frac{1}{2} - t\right)$. $\sin^2(\pi t)$ does not interfere with the residues, for it equals 1 at half integers, so it can be ignored. Then the right-hand side is the integrand of a $G_{4,4}^{2,4}$. 
Proof of Theorem (2), first equality

Nesterenko’s theorem connects $G_{4,4}^{2,4}$ to a triple integral. The entries in the $G_{4,4}^{2,4}$ need to satisfy special properties. In particular,
Proof of Theorem (2), first equality

Nesterenko’s theorem connects $G_{4,4}^{2,4}$ to a triple integral. The entries in the $G_{4,4}^{2,4}$ need to satisfy special properties. In particular,

$$a(z) := G_{4,4}^{2,2} \left( \begin{array}{c}
0,1,1,1 \\
-\frac{1}{2},1,\frac{1}{2},-\frac{1}{2}
\end{array} | z \right)$$

does not satisfy these properties. But $a(1) = -2\pi W_4(1)$. 

Proof of Theorem (2), first equality

Nesterenko’s theorem connects $G_{4,4}^{2,4}$ to a triple integral. The entries in the $G_{4,4}^{2,4}$ need to satisfy special properties. In particular, $a(z) := G_{4,4}^{2,2} \left( \begin{array}{c} 0,1,1,1 \\ -\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2} \end{array} | z \right)$ does not satisfy these properties. But $a(1) = -2\pi W_4(1)$.

However, $c := -G_{4,4}^{2,2} \left( \begin{array}{c} 0,1,1,1 \\ \frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2} \end{array} | 1 \right)$ does. Experimentally we observed $a(1) = 4c$. 
Proof of Theorem (2), first equality

*Nesterenko’s theorem* connects $G_{4,4}^{2,4}$ to a triple integral. The entries in the $G_{4,4}^{2,4}$ need to satisfy special properties. In particular,

$$a(z) := G_{4,4}^{2,2} \left( \begin{array}{c} 0,1,1,1 \\ -\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2} \end{array} \right| z \right)$$

does not satisfy these properties.

But $a(1) = -2\pi W_4(1)$.

However, $c := -G_{4,4}^{2,2} \left( \begin{array}{c} 0,1,1,1 \\ \frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2} \end{array} \right| 1 \right)$ does. Experimentally we observed $a(1) = 4c$.

We use these easy identities:

$$\frac{d}{dz} \left( z^{-b_1} G_{4,4}^{2,2} \left( \begin{array}{c} a_1, a_2, a_3, a_4 \\ b_1, b_2, b_3, b_4 \end{array} \right| z \right) \right) = \frac{-1}{z^{1+b_1}} G_{4,4}^{2,2} \left( \begin{array}{c} a_1, a_2, a_3, a_4 \\ b_1 + 1, b_2, b_3, b_4 \end{array} \right| z \right)$$

$$\frac{d}{dz} \left( z^{1-a_1} G_{4,4}^{2,2} \left( \begin{array}{c} a_1, a_2, a_3, a_4 \\ b_1, b_2, b_3, b_4 \end{array} \right| z \right) \right) = \frac{1}{z^{a_1}} G_{4,4}^{2,2} \left( \begin{array}{c} a_1 - 1, a_2, a_3, a_4 \\ b_1, b_2, b_3, b_4 \end{array} \right| z \right)$$
Applying the first identity to $a(z)$ and using the \textit{product rule}, we get $\frac{1}{2}a(1) + a'(1) = c$. 
Applying the first identity to $a(z)$ and using the *product rule*, we get $\frac{1}{2}a(1) + a'(1) = c$.

Applying the second identity to $a(z)$, we obtain $a'(1) = -c$ after simplifications. Hence $a(1) = 4c$. 

Using Nesterenko's theorem: 

$$W_4(1) = \frac{4\pi}{3} \int_0^1 \int_0^1 \int_0^1 \int_0^1 \sqrt{x} (1 - y)(1 - z)(1 - x) yz \, dx \, dy \, dz.$$ 

Change of variable $z' = 1 - z$, then use $z'^{1/2} = (z')^{1/2} - (1 - (1 - z'))$ to split it into two integrals. Each integral satisfies Zudilin's theorem, which converts such integrals into $7F6$'s.
Applying the first identity to \( a(z) \) and using the *product rule*, we get \( \frac{1}{2}a(1) + a'(1) = c \).

Applying the second identity to \( a(z) \), we obtain \( a'(1) = -c \) after simplifications. Hence \( a(1) = 4c \).

Using Nesterenko’s theorem:

\[
W_4(1) = \frac{4}{\pi^3} \int_0^1 \int_0^1 \int_0^1 \sqrt{\frac{x(1-y)(1-z)}{(1-x)yz(1-x(1-yz))}} \, dx \, dy \, dz.
\]
Applying the first identity to \( a(z) \) and using the *product rule*, we get \( \frac{1}{2}a(1) + a'(1) = c \).

Applying the second identity to \( a(z) \), we obtain \( a'(1) = -c \) after simplifications. Hence \( a(1) = 4c \).

Using Nesterenko’s theorem:

\[
W_4(1) = \frac{4}{\pi^3} \int_0^1 \int_0^1 \int_0^1 \sqrt{\frac{x(1-y)(1-z)}{(1-x)yz(1-x(1-yz))}} \, dx \, dy \, dz.
\]

Change of variable \( z' = 1 - z \), then use
\[
(z')^{\frac{1}{2}} = (z')^{-\frac{1}{2}} (1 - (1 - z')) = (z')^{-\frac{1}{2}} - (z')^{-\frac{1}{2}} (1 - z')
\]
to split it into two integrals.
Applying the first identity to $a(z)$ and using the product rule, we get $\frac{1}{2}a(1) + a'(1) = c$.

Applying the second identity to $a(z)$, we obtain $a'(1) = -c$ after simplifications. Hence $a(1) = 4c$.

Using Nesterenko’s theorem:

$$W_4(1) = \frac{4}{\pi^3} \int_0^1 \int_0^1 \int_0^1 \sqrt{\frac{x(1-y)(1-z)}{(1-x)yz(1-x(1-yz))}} \, dx \, dy \, dz.$$ 

Change of variable $z' = 1 - z$, then use $(z')^{\frac{1}{2}} = (z')^{-\frac{1}{2}}(1 - (1 - z')) = (z')^{-\frac{1}{2}} - (z')^{-\frac{1}{2}}(1 - z')$ to split it into two integrals.

Each integral satisfies Zudilin’s theorem, which converts such integrals into $\, \, \, 7F_{6}$’s.
Proof of Theorem (2), second equality

We convert all hypergeometric terms into triple integrals, judiciously using Bailey’s identity, Nesterenko’s theorem, and Zudilin’s theorem (for they produce multiple equivalent forms).
Proof of Theorem (2), second equality

We convert all hypergeometric terms into triple integrals, judiciously using Bailey’s identity, Nesterenko’s theorem, and Zudilin’s theorem (for they produce multiple equivalent forms).

We also use Zudilin’s result which gives a non-trivial permutation of the exponents of $x, y, z$ in the triple integral, while leaving its value unchanged.
Proof of Theorem (2), second equality

We convert all hypergeometric terms into triple integrals, judiciously using Bailey’s identity, Nesterenko’s theorem, and Zudilin’s theorem (for they produce multiple equivalent forms).

We also use Zudilin’s result which gives a non-trivial permutation of the exponents of $x, y, z$ in the triple integral, while leaving its value unchanged.

These integrals are nice enough that they can be reduced to 1D integrals of $E$ and $K$. 
Proof of Theorem (2), second equality

We convert all hypergeometric terms into triple integrals, judiciously using Bailey’s identity, Nesterenko’s theorem, and Zudilin’s theorem (for they produce multiple equivalent forms).

We also use Zudilin’s result which gives a non-trivial permutation of the exponents of $x, y, z$ in the triple integral, while leaving its value unchanged.

These integrals are nice enough that they can be reduced to 1D integrals of $E$ and $K$.

When we pick the “right” integrals, the integrands (as functions of $E$ and $K$) on both sides equal.
Outline

1. Introduction

2. Expectations
   - Experimental maths 1

3. Densities

4. 3 and 4 steps
   - Experimental maths 2
For $n \geq 4$,

$$p_n(t) = \int_0^\infty xt J_0(xt) J_n^0(x) \, dx$$
For $n \geq 4$,
\[ p_n(t) = \int_0^\infty x t J_0(xt) J_n^0(x) dx \]

So probability of returning to the unit disk is
\[ \int_0^1 p_n(t) dt = \int_0^\infty J_1(x) J_n^0(x) dx = \left[ \frac{-J_0(x)^{n+1}}{n+1} \right]_0^\infty = \frac{1}{n+1}. \]
For $n \geq 4$,

$$p_n(t) = \int_0^\infty xt J_0(xt) J^n_0(x) dx$$

So probability of returning to the unit disk is

$$\int_0^1 p_n(t) dt = \int_0^\infty J_1(x) J^n_0(x) dx = \left[ \frac{-J_0(x)^{n+1}}{n+1} \right]_0^\infty = \frac{1}{n+1}.$$

For $n = 2$ and $3$ the probability is elementary.
For \( n \geq 4 \),
\[
p_n(t) = \int_0^\infty xtJ_0(xt)J_0^n(x)dx
\]

So probability of returning to the unit disk is
\[
\int_0^1 p_n(t)dt = \int_0^\infty J_1(x)J_0^n(x)dx = \left[\frac{-J_0(x)^{n+1}}{n+1}\right]_0^\infty = \frac{1}{n+1}.
\]

For \( n = 2 \) and \( 3 \) the probability is elementary.

\( p_n \) is smooth for \( n \geq 6 \).
Our definition of $p_n$ takes advantage of radial symmetry. A true 2D probability density $\psi_n$ requires

$$W_n(s) = \int_0^n \psi_n(x)x^s \ 2\pi x \, dx.$$ 

That is, $p_n(x) = 2\pi x \psi_n(x)$. 

Lord Rayleigh
Our definition of $p_n$ takes advantage of radial symmetry. A true 2D probability density $\psi_n$ requires

$$W_n(s) = \int_0^n \psi_n(x)x^s 2\pi x dx.$$  

That is, $p_n(x) = 2\pi x\psi_n(x)$.

Rayleigh gave approximate $\psi_n$ for large $n$, first by approximating the problem in 1D using the central limit theorem (for Bernoulli trials: $\frac{1}{\sqrt{n\pi/2}}e^{-2x^2/n}$).
Lord Rayleigh

- Our definition of $p_n$ takes advantage of radial symmetry. A true 2D probability density $\psi_n$ requires

$$W_n(s) = \int_0^n \psi_n(x)x^s 2\pi x dx.$$ 

That is, $p_n(x) = 2\pi x\psi_n(x)$.

- Rayleigh gave approximate $\psi_n$ for large $n$, first by approximating the problem in 1D using the central limit theorem (for Bernoulli trials: $\frac{1}{\sqrt{n\pi/2}} e^{-2x^2/n}$).

- He then allowed the walks to be on a lattice, finally relaxing it to the plane, modifying his approximation.
Our definition of $p_n$ takes advantage of radial symmetry. A true 2D probability density $\psi_n$ requires

$$W_n(s) = \int_0^n \psi_n(x) x^s 2\pi x dx.$$ 

That is, $p_n(x) = 2\pi x \psi_n(x)$.

Rayleigh gave approximate $\psi_n$ for large $n$, first by approximating the problem in 1D using the central limit theorem (for Bernoulli trials: $e^{-2x^2/n}$).

He then allowed the walks to be on a lattice, finally relaxing it to the plane, modifying his approximation.

$\psi_n(x) \approx \frac{1}{n\pi} e^{-x^2/n}$, like a 2D central limit theorem.
Our definition of $p_n$ takes advantage of radial symmetry. A true 2D probability density $\psi_n$ requires

$$W_n(s) = \int_0^n \psi_n(x)x^s 2\pi x dx.$$ 

That is, $p_n(x) = 2\pi x \psi_n(x)$.

Rayleigh gave approximate $\psi_n$ for large $n$, first by approximating the problem in 1D using the central limit theorem (for Bernoulli trials: $\frac{1}{\sqrt{n\pi/2}}e^{-2x^2/n}$).

He then allowed the walks to be on a lattice, finally relaxing it to the plane, modifying his approximation.

$\psi_n(x) \approx \frac{1}{n\pi}e^{-x^2/n}$, like a 2D central limit theorem.

This is very accurate even for moderate $n$. 
$p_n$ with approximations superimposed.

A better approximation is $xe^{-x^2/n} \left( \frac{4n^3-2n^2+4nx^2-x^4}{2n^4} \right)$. 
Recursion for $W_n$

We condition the distance $z$ of an $(n + m)$-step walk on $x$ (first $n$ steps), followed by $y$ (remaining $m$ steps).
Recursion for $W_n$

We condition the distance $z$ of an $(n + m)$-step walk on $x$ (first $n$ steps), followed by $y$ (remaining $m$ steps).

By the cosine rule,

$$z^2 = x^2 + y^2 + 2xy \cos(\theta).$$
Recursion for $W_n$

We condition the distance $z$ of an $(n + m)$-step walk on $x$ (first $n$ steps), followed by $y$ (remaining $m$ steps).

By the cosine rule,

$$z^2 = x^2 + y^2 + 2xy \cos(\theta).$$

The moments are worked out by CAS:

$$g_s(x, y) := \frac{1}{\pi} \int_0^\pi z^s \, d\theta = y^s \, \text{Re} \, \, _2F_1 \left( -\frac{s}{2}, -\frac{s}{2} \mid \frac{x^2}{y^2} \right).$$
Recursion for $W_n$

We condition the distance $z$ of an $(n + m)$-step walk on $x$ (first $n$ steps), followed by $y$ (remaining $m$ steps).

By the cosine rule,

$$z^2 = x^2 + y^2 + 2xy \cos(\theta).$$

The moments are worked out by CAS:

$$g_s(x, y) := \frac{1}{\pi} \int_0^\pi z^s \, d\theta = y^s \text{ Re } \text{Re}_1 \left( -\frac{s}{2}, -\frac{s}{2} \mid \frac{x^2}{y^2} \right).$$

Therefore $W_{n+m}(s) = \int_0^n \int_0^m g_s(x, y) p_n(x)p_m(y)dx\,dy$.  \hspace{1cm} (1)
Recursion for $\psi_n$

Let $r$ be the position vector after $n$ steps, and $s$ be the position vector of the $n$th step.
Recursion for $\psi_n$

Let $r$ be the position vector after $n$ steps, and $s$ be the position vector of the $n$th step.

Then, upon using polar coordinates and the cosine rule,

$$\psi_n(r) = \int \frac{\delta_1(|s|)}{2\pi} \psi_{n-1}(|r-s|) ds = \int_0^{2\pi} \psi_{n-1}(\sqrt{r^2 + 1 - 2r \cos t}) \frac{dt}{2\pi}.$$

Combined with $\psi_2$, this gives

$$\rho_3(x) = \sqrt{x} \pi \Re K_0(\sqrt{x+1})^{\frac{3}{4} \left(3 - x\right)}.$$
Recursion for $\psi_n$

Let $r$ be the position vector after $n$ steps, and $s$ be the position vector of the $n$th step.

Then, upon using polar coordinates and the cosine rule,

$$\psi_n(r) = \int \frac{\delta_1(|s|)}{2\pi} \psi_{n-1}(|r-s|) ds = \int_0^{2\pi} \frac{\psi_{n-1}(\sqrt{r^2 + 1 - 2r \cos t})}{2\pi} dt.$$

Combined with $\psi_2$, this gives

$$p_3(x) = \frac{\sqrt{x}}{\pi^2} \text{Re} \ K \left( \sqrt{\frac{(x + 1)^3(3 - x)}{16x}} \right).$$
Recursion for $\psi_n$

Let $r$ be the position vector after $n$ steps, and $s$ be the position vector of the $n$th step.

Then, upon using polar coordinates and the cosine rule,

$$\psi_n(r) = \int \frac{\delta_1(|s|)}{2\pi} \psi_{n-1}(|r-s|) ds = \int_0^{2\pi} \psi_{n-1}\left(\sqrt{r^2 + 1 - 2r \cos t}\right) \frac{dt}{2\pi}.$$ 

Combined with $\psi_2$, this gives

$$p_3(x) = \frac{\sqrt{x}}{\pi^2} \text{Re} \ K\left(\sqrt{\frac{(x + 1)^3(3 - x)}{16x}}\right).$$

Put $r = 0$, we get $\psi_n(0) = \psi_{n-1}(1)$.
Recursion for $\psi_n$

Let $r$ be the position vector after $n$ steps, and $s$ be the position vector of the $n$th step.

Then, upon using polar coordinates and the cosine rule,

$$\psi_n(r) = \int \frac{\delta_1(|s|)}{2\pi} \psi_{n-1}(|r-s|)ds = \int_0^{2\pi} \frac{\psi_{n-1}(\sqrt{r^2 + 1 - 2r \cos t})}{2\pi} dt.$$

Combined with $\psi_2$, this gives

$$p_3(x) = \frac{\sqrt{x}}{\pi^2} \operatorname{Re} K \left( \frac{\sqrt{(x + 1)^3(3 - x)}}{16x} \right).$$

Put $r = 0$, we get $\psi_n(0) = \psi_{n-1}(1) = \frac{p_{n-1}(1)}{2\pi} = \frac{p_n'(0)}{2\pi} = \frac{\text{Res}_{-2} W_n}{2\pi}$. 
Alternative form for $p_n$

We now use the *sine rule* to make a change variable, so the last integral in (1) becomes $dz$ instead of $dy$:

$$W_{n+m}(s) = \int_0^{n+m} z^s \left\{ \int_0^n \int_0^\pi z p_n(x)p_m(y) dtdx \right\} dz,$$

where $y = \sqrt{x^2 + z^2 - 2xz \cos t}$. 
Alternative form for $p_n$

We now use the *sine rule* to make a change variable, so the last integral in (1) becomes $dz$ instead of $dy$:

$$W_{n+m}(s) = \int_0^{n+m} z^s \left\{ \int_0^n \int_0^\pi \frac{z}{\pi y} p_n(x)p_m(y) dt dx \right\} dz,$$

where $y = \sqrt{x^2 + z^2 - 2xz \cos t}$.

By uniqueness, the expression inside the braces is $p_{n+m}$. 
We now use the sine rule to make a change variable, so the last integral in (1) becomes \(dz\) instead of \(dy\):

\[
W_{n+m}(s) = \int_0^{n+m} z^s \left\{ \int_0^n \int_0^\pi \frac{z}{\pi y} p_n(x)p_m(y) dt dx \right\} dz,
\]

where \(y = \sqrt{x^2 + z^2 - 2xz \cos t}\).

By uniqueness, the expression inside the braces is \(p_{n+m}\).

Combined with \(p_2\), we have

\[
p_4(t) = \frac{8t}{\pi^3} \int_0^2 \text{Re} \left( \frac{K \left( \sqrt{\frac{16xt}{(x+t)^2(4-(x-t)^2)}} \right)}{\sqrt{(x+t)^2(4-(x-t)^2)}} \right) \frac{dx}{\sqrt{4-x^2}},
\]

which is better numerically than its Bessel counterpart.
Outline

1 Introduction

2 Expectations
   • Experimental maths 1

3 Densities

4 3 and 4 steps
   • Experimental maths 2
Poles of $W_3$ via $p_3$

In $p_3$, we have

$$K \left( \sqrt{\frac{16x^3}{(3-x)^3(1+x)}} \right) = \frac{3-x}{3+3x} K \left( \sqrt{\frac{16x}{(3-x)(1+x)^3}} \right),$$

as both sides satisfy the same differential equation.
Poles of $W_3$ via $p_3$

In $p_3$, we have $K\left(\sqrt{\frac{16x^3}{(3-x)^3(1+x)}}\right) = \frac{3-x}{3+3x} K\left(\sqrt{\frac{16x}{(3-x)(1+x)^3}}\right)$, as both sides satisfy the same differential equation.

So we can write $p_3$ cleanly in terms of the AGM, enabling us to use a result of Borwein et al. So on $[0, 1)$

$$p_3(x) = \frac{2}{\sqrt{3\pi}} x \sum_{k=0}^{\infty} W_3(2k) \left(\frac{x}{3}\right)^{2k}.$$
Poles of $W_3$ via $p_3$

In $p_3$, we have $K \left( \sqrt{\frac{16x^3}{(3-x)^3(1+x)}} \right) = \frac{3-x}{3+3x} K \left( \sqrt{\frac{16x}{(3-x)(1+x)^3}} \right)$, as both sides satisfy the same differential equation.

So we can write $p_3$ cleanly in terms of the AGM, enabling us to use a result of Borwein et al. So on $[0, 1)$

$$p_3(x) = \frac{2}{\sqrt{3\pi}} x \sum_{k=0}^{\infty} W_3(2k) \left( \frac{x}{3} \right)^{2k}.$$  

Using this series, we compute (with lots of care), for small $a > 0$,

$$\int_{0}^{a} p_3(x) x^s \, dx = \frac{2a^{s+2}}{\sqrt{3\pi}(s+2)} + \frac{2a^{s+4}}{3\sqrt{3\pi}(s+4)} + \cdots$$

so the residues of $W_3$ can be read off, namely,

$$\text{Res}_{(-2k-2)} W_3 = \frac{2}{\pi \sqrt{3}} \frac{W_3(2k)}{9^k}.$$
Poles of $W_3$ via $p_3$

In $p_3$, we have $K \left( \sqrt{\frac{16x^3}{(3-x)^3(1+x)}} \right) = \frac{3-x}{3+3x} \left( \sqrt{\frac{16x}{(3-x)(1+x)^3}} \right)$, as both sides satisfy the same differential equation.

So we can write $p_3$ cleanly in terms of the AGM, enabling us to use a result of Borwein et al. So on $[0, 1)$

$$p_3(x) = \frac{2}{\sqrt{3\pi}} x \sum_{k=0}^{\infty} W_3(2k) \left( \frac{x}{3} \right)^{2k}.$$  

Using this series, we compute (with lots of care), for small $a > 0$,

$$\int_0^{a} p_3(x)x^s dx = \frac{2a^{s+2}}{\sqrt{3\pi} (s + 2)} + \frac{2a^{s+4}}{3\sqrt{3\pi} (s + 4)} + \cdots$$

so the residues of $W_3$ can be read off, namely,

$$\text{Res}_{(-2k-2)} W_3 = \frac{2}{\pi \sqrt{3}} \frac{W_3(2k)}{9^k}.$$  

But if $p_4$ admits a similar series, how can this reconcile with the double poles of $W_4$?
As $\text{Re } K(x) = \frac{1}{x} K \left( \frac{1}{x} \right)$ for $x > 1$, we split $p_3$ over $[0, 1]$ and $[1, 3]$, obtaining $W_3(-1) = \int_0^3 \frac{p_3(x)}{x} \, dx =$

$$\frac{4}{\pi^2} \int_0^1 K \left( \frac{\sqrt{\frac{16x}{(3-x)(1+x)^3}}}{16x} \right) \, dx + \frac{1}{\pi^2} \int_1^3 K \left( \frac{\sqrt{(3-x)(1+x)^3}}{16x} \right) \, dx.$$
Introduction \hfill Expectations \hfill Densities \hfill 3 and 4 steps \hfill Experimental maths 2

**Functional equation for \( p_3 \)**

As \( \text{Re } K(x) = \frac{1}{x} K \left( \frac{1}{x} \right) \) for \( x > 1 \), we split \( p_3 \) over \([0, 1]\) and \([1, 3]\), obtaining \( W_3(-1) = \int_0^3 \frac{p_3(x)}{x} \, dx = \)

\[
\frac{4}{\pi^2} \int_0^1 \frac{K \left( \sqrt{\frac{16x}{(3-x)(1+x)^3}} \right)}{\sqrt{(3-x)(1+x)^3}} \, dx + \frac{1}{\pi^2} \int_1^3 \frac{K \left( \sqrt{\frac{(3-x)(1+x)^3}{16x}} \right)}{\sqrt{x}} \, dx.
\]

Numerically we noted the two integrals equal. Proof: change of variable \( x \rightarrow \frac{3-t}{1+t} \) in the second integral.
Functional equation for $p_3$

As $\text{Re } K(x) = \frac{1}{x} K \left( \frac{1}{x} \right)$ for $x > 1$, we split $p_3$ over $[0, 1]$ and $[1, 3]$, obtaining $W_3(-1) = \int_0^3 \frac{p_3(x)}{x} \, dx =$

$$\frac{4}{\pi^2} \int_0^1 K \left( \sqrt{\frac{16x}{(3-x)(1+x)^3}} \right) \frac{1}{\sqrt{(3-x)(1+x)^3}} \, dx + \frac{1}{\pi^2} \int_1^3 K \left( \sqrt{\frac{(3-x)(1+x)^3}{16x}} \right) \frac{1}{\sqrt{x}} \, dx.$$

Numerically we noted the two integrals equal. Proof: change of variable $x \to \frac{3-t}{1+t}$ in the second integral.

This leads to a modular property: with the involution $\sigma(x) = \frac{3-x}{1+x}$,

$$p_3(x) = \frac{4x}{(3-x)(x+1)} p_3(\sigma(x)).$$
Functional equation for $p_3$

As $\text{Re } K(x) = \frac{1}{x}K\left(\frac{1}{x}\right)$ for $x > 1$, we split $p_3$ over $[0, 1]$ and $[1, 3]$, obtaining $W_3(-1) = \int_0^3 \frac{p_3(x)}{x} \, dx = \frac{4}{\pi^2} \int_0^1 \frac{K\left(\sqrt{\frac{16x}{(3-x)(1+x)^3}}\right)}{\sqrt{(3-x)(1+x)^3}} \, dx + \frac{1}{\pi^2} \int_1^3 \frac{K\left(\sqrt{\frac{(3-x)(1+x)^3}{16x}}\right)}{\sqrt{x}} \, dx$.

Numerically we noted the two integrals equal. Proof: change of variable $x \rightarrow \frac{3-t}{1+t}$ in the second integral.

This leads to a modular property: with the involution $\sigma(x) = \frac{3-x}{1+x}$,

$$p_3(x) = \frac{4x}{(3-x)(x+1)} p_3(\sigma(x)).$$

Also, $W_3(-1) = \frac{4}{\sqrt{3}\pi} \sum_{k=0}^{\infty} \frac{W_3(2k)}{9^k(2k+1)}$. 
Series for $p_4$

Jon asked us to plot $p_4'(x)$ for small $x$. Armin correctly used the true formula,

$$\lim_{h \to 0} \frac{p_4(x + h) - p_4(x)}{h},$$

I, however, foolishly used the "formula",

$$\lim_{h \to 0} \frac{p_4(x + h) - p_4(h)}{x}.$$

Amazingly, we produced almost the same plot, except mine was vertically translated up by $\approx 0.14$.

Unfazed by my failure to find a derivative from first principles, this means, very nearly, $p_4$ satisfies the differential equation

$$f'(x) + ax = f(x),$$

which even I can solve:

$$f(x) = bx - ax \log x,$$

where $b \approx 0.33$. Indeed, we have

$$\int_0^1 f(x) \, dx = \frac{1}{15}.21/26.$$
Series for $p_4$

Jon asked us to plot $p'_4(x)$ for small $x$. Armin correctly used the true formula,

$$\lim_{h \to 0} \frac{p_4(x + h) - p_4(x)}{h},$$

I, however, foolishly used the “formula”,

$$\lim_{h \to 0} \frac{p_4(x + h) - p_4(h)}{x}. $$
Jon asked us to plot $p_4'(x)$ for small $x$. Armin correctly used the true formula,

$$\lim_{h \to 0} \frac{p_4(x + h) - p_4(x)}{h},$$

I, however, foolishly used the “formula”,

$$\lim_{h \to 0} \frac{p_4(x + h) - p_4(h)}{x}.$$

Amazingly, we produced almost the same plot, except mine was vertically translated up by $a \approx 0.14$. 
Series for $p_4$

Jon asked us to plot $p'_4(x)$ for small $x$. Armin correctly used the true formula,

$$\lim_{h \to 0} \frac{p_4(x + h) - p_4(x)}{h},$$

I, however, foolishly used the “formula”,

$$\lim_{h \to 0} \frac{p_4(x + h) - p_4(h)}{x}.$$

Amazingly, we produced almost the same plot, except mine was vertically translated up by $a \approx 0.14$.

Unfazed by my failure to find a derivative from first principles, this means, very nearly, $p_4$ satisfies the differential equation

$$f'(x) + a = \frac{f(x)}{x},$$

which even I can solve: $f(x) = bx - ax \log x$, where $b \approx 0.33$ as $\int_0^1 f(x)dx = \frac{1}{5}$. 
This explains the double pole!
This explains the double pole!

In fact, if the series were to be consistent with the residues and coefficients of the double pole, then we must have:

\[ p_4(x) = \sum_{n=1}^{\infty} \left( a_4(n) - r_4(n) \log x \right) x^{2n-1}, \]

where \( a_4(n) \) are the residues at \(-2n\) and \( r_4(n) \) are the coefficients of the double pole at \(-2n\).
This explains the double pole!

In fact, if the series were to be consistent with the residues and coefficients of the double pole, then we must have:

\[
p_4(x) = \sum_{n=1}^{\infty} \left( a_4(n) - r_4(n) \log x \right) x^{2n-1},
\]

where \( a_4(n) \) are the residues at \(-2n\) and \( r_4(n) \) are the coefficients of the double pole at \(-2n\).

The first approximation is

\[
\left( \frac{9 \log 2}{2\pi^2} - \frac{3}{2\pi^2} \log x \right) x.
\]

\( r_4(n) \) may be obtained in closed form by recursion.
$p_4$ versus conjectured expansion on $[0, 2]$. 

![Graph showing $p_4$ versus conjectured expansion on $[0, 2]$.](image)
$p_4$ versus conjectured expansion on $[0, 2]$.

Like $p_3$, $p_4$ also has a clean AGM form.
\( p_4 \) versus conjectured expansion on \([0, 2]\).

Like \( p_3 \), \( p_4 \) also has a clean AGM form.

\( p_4 \) can also be written in terms of the Domb numbers,

\[
W_4(2n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k} \binom{2n - 2k}{n - k}.
\]
From our series for $p_3$, Zudilin (using modular tools) deduced the closed form

$$p_3(x) = \frac{2\sqrt{3}x}{\pi(3 + x^2)} \ {}_2F_1\left(\frac{1}{3}, \frac{2}{3} \left| \frac{x^2(9 - x^2)^2}{(3 + x^2)^3} \right.\right),$$
Closed forms

From our series for $p_3$, Zudilin (using modular tools) deduced the closed form

$$p_3(x) = \frac{2\sqrt{3}x}{\pi(3 + x^2)} \, 2F_1 \left( \frac{1}{3}, \frac{2}{3} \middle| \frac{x^2(9 - x^2)^2}{(3 + x^2)^3} \right),$$

as well as a closed formed for $p_4$ on $[2, 4]$:

$$p_4(x) = \frac{2\sqrt{16 - x^2}}{\pi^2 x} \, 3F_2 \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \middle| \frac{(16 - x^2)^3}{108x^4} \right).$$

Numerically, this works on $[0, 4]$ by taking the real part.
Closed forms

From our series for $p_3$, Zudilin (using modular tools) deduced the closed form

$$p_3(x) = \frac{2\sqrt{3}x}{\pi(3 + x^2)} \, _2F_1 \left( \begin{array}{c} \frac{1}{3}, \frac{2}{3} \\ 1 \end{array} \bigg| \frac{x^2(9 - x^2)^2}{(3 + x^2)^3} \right),$$

as well as a closed formed for $p_4$ on $[2, 4]$:

$$p_4(x) = \frac{2\sqrt{16 - x^2}}{\pi^2x} \, _3F_2 \left( \begin{array}{c} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ \frac{5}{6}, \frac{7}{6} \end{array} \bigg| \frac{(16 - x^2)^3}{108x^4} \right).$$

Numerically, this works on $[0, 4]$ by taking the real part.

We get eerie connections with $W_3(s)$, for instance

$$p_4(2) = \frac{\sqrt{3}}{\pi} W_3(-1) \quad \text{and} \quad p_3(\sqrt{3})^2 = 4p_3(2\sqrt{3} - 3)^2 = \frac{3}{2\pi^2} W_3(-1).$$
Future work

- Prove expansion for $p_4$, and prove closed form on all of $[0, 4]$. 
Future work

- Prove expansion for $p_4$, and prove closed form on all of $[0, 4]$.
- Other properties of $p_4$, for instance any functional equations, or points of inflection.
Future work

- Prove expansion for $p_4$, and prove closed form on all of $[0, 4]$.
- Other properties of $p_4$, for instance any functional equations, or points of inflection.
- Properties of $W_5$ and $p_5$, for example, why is $p_5$ almost linear on $[0, 1]$?
Future work

- Prove expansion for $p_4$, and prove closed form on all of $[0, 4]$.
- Other properties of $p_4$, for instance any functional equations, or points of inflection.
- Properties of $W_5$ and $p_5$, for example, why is $p_5$ almost linear on $[0, 1]$?
- Links to Calabi-Yau differential equations?
Future work

- Prove expansion for $p_4$, and prove closed form on all of $[0, 4]$.
- Other properties of $p_4$, for instance any functional equations, or points of inflection.
- Properties of $W_5$ and $p_5$, for example, why is $p_5$ almost linear on $[0, 1]$?
- Links to Calabi-Yau differential equations?
- More closed forms for derivatives and residues for $W_3$ and $W_4$. 
Thank you!
Thank you!

- Comments?
- Questions?
- Criticisms?