Simple Derivation of Quantum Scaling in Child–Langmuir Law

L. K. Ang, *Member, IEEE*, Y. Y. Lau, *Member, IEEE*, and T. J. T. Kwan

***Abstract*—A simple derivation of the new scaling of Child–Langmuir law in the quantum regime is presented. Based on a dimensional argument of the Schrodinger equation and the Poisson equation, the limiting current in the deeply quantum regime is found to be proportional to the square root of the gap voltage and to the inverse fourth power of gap spacing. The importance of electron exchange-correlation interactions in the quantum regime is discussed.**

***Index Terms*—Beams, cathodes, nanotechnology, space charge.**

I. Introduction

The study of space-charge limited current began almost a century ago [1], [13] and continues to be an active field of research [2]–[12]. In the one-dimensional (1-D) planar geometry with zero electron emission energy, the classical Child–Langmuir (CL) law [1], [13] gives the maximum current density (J_{CL}) allowed for steady-state electron beam transport across a gap of gap spacing D and gap voltage V_g

$$J_{CL} = \frac{4}{9}\varepsilon_0 \sqrt{\frac{2e}{m}} \frac{V_g^{\frac{3}{2}}}{D^2} \tag{1}$$

where e and m are the electron's charge and mass, respectively, and ε_0 is the free-space permittivity. The extension of the classical 1-D CL law to a multidimensional model is a subject of much interest in recent years [2]–[9].

The recent advances in the fabrication of nano-scale structures, such as nano diode and nano gap, stimulated the development of the CL law in the nanometer regime, where a quantum mechanical treatment may be necessary [10]–[12]. The recent quantum model [12], which includes the effects of electron tunneling, electron space charge, and electron exchange correlation (using local density approximation), predicts that the classical value in the CL law can be increased significantly. The limiting current density in the quantum regime (J_Q) may be written in the general form

$$J_Q = \mu_Q \times J_{CL} \tag{2}$$

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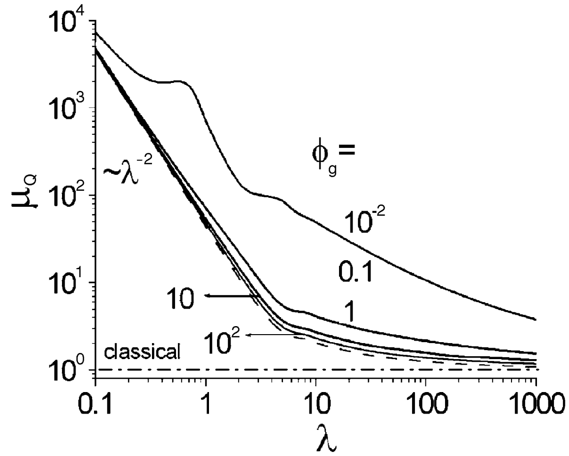

Fig. 1. Calculated quantum enhancement factor μ_Q as a function of λ for various $\phi_g = 10^{-2}$ to 10^2 (solid lines), $\phi_g \gg 1$ (dashed line: without exchange-correlation effects), and classical limit (dashed-dotted line). From Fig. 1 in [12].

where μ_Q is the quantum enhancement factor, which depends on the gap voltage, gap spacing, and electron emission energy [12]. Fig. 1 shows the calculated values of μ_Q, at various gap voltage V_g (or ϕ_g, when normalized to the Hartree voltage $e/4\pi\varepsilon_0 a_0 = 27.2$ V, where $a_0 = 0.0529$ nm is the Bohr radius) and zero emission energy level, as a function of $\lambda = D/\sqrt{\hbar^2/2\,\mathrm{em}V_g}$, where λ is the ratio of gap spacing to the electron De Broglie wavelength at V_g.

The model indicates that the well-known scaling of $V_g^{3/2}$ and D^{-2} [cf., (1)] is no longer valid in the quantum regime, and a new scaling of $V_g^{1/2}$ and D^{-4} is established from the numerical computations. In the limit $\lambda \ll 1$, μ_Q is proportional to λ^{-2} as shown in Fig. 1, and (2) becomes

$$J_Q \propto \frac{\varepsilon_0 \hbar^2}{e^{\frac{1}{2}} m^{\frac{3}{2}}} \frac{V_g^{\frac{1}{2}}}{D^4}. \tag{3}$$

While this new scaling was first explicitly pointed out in [12], its dependence on energy and on gap spacing was also implicitly shown in the earlier model [10], where the electron exchange-correlation had been ignored. In this paper, we will present a simple derivation of (3), the new quantum scaling.

II. Derivation of Quantum Scaling Law

From our quantum models [12], we find that the quantum scaling law shown in (3) is roughly the same for calculations with or without the effects of electron exchange correlation. Thus, we construct this quantum scaling from a dimensional

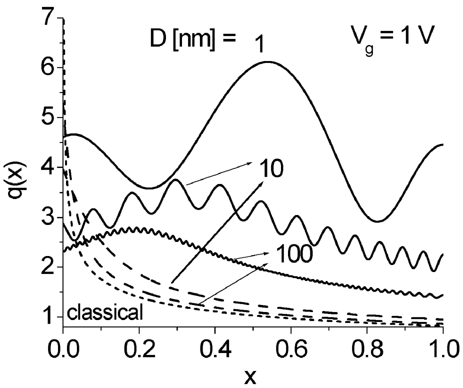

Fig. 2. Calculated normalized electron wave amplitude for $D = 1, 10$, and 100 nm at $V_g = 1$ V. Two dashed lines are without including the exchange-correlation potential, and the short dashed line is the classical limit.

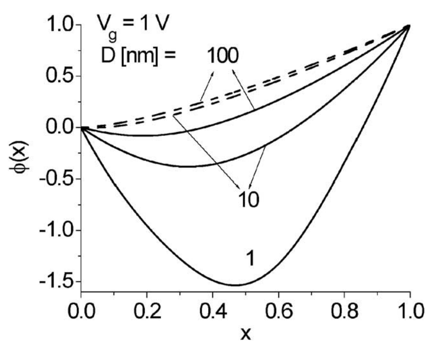

Fig. 3. Calculated normalized electrical potential for $D = 1, 10$, and 100 nm at $V_g = 1$ V. Two dashed lines are without including the exchange-correlation potential.

argument using the Schrodinger equation and Poisson equation, without including the exchange correlation component

$$\frac{\hbar^2}{2m}\frac{d^2\Psi}{dx^2} = eV\Psi \tag{4}$$

$$\frac{d^2V}{dx^2} = \frac{en}{\varepsilon_0} \tag{5}$$

where ψ is the electron wave function, V is the electrostatic potential, n is the electron density, and the electron energy is assumed to be zero. Equation (5) gives dimensionally

$$V = \frac{en}{\varepsilon_0}x^2. \tag{6}$$

Substituting (6) into (4) gives, dimensionally

$$n = \frac{\hbar^2\varepsilon_0}{2e^2m}\frac{1}{x^4} \tag{7}$$

which becomes the electron density scale n_s, introduced in [10] by setting $x = D$.

The current density scale is $J = env = en\sqrt{2\text{ eV/m}}$. Using (7), we have

$$J = \frac{\varepsilon_0\hbar^2}{e^{\frac{1}{2}}m^{\frac{3}{2}}}\frac{V^{\frac{1}{2}}}{x^4} \tag{8}$$

which reduces to (3) by setting $V = V_g$ and $x = D$.

Of course, the classical CL scaling [see (1)] may similarly be constructed from a dimensional argument. The Poisson equation (6) yields $n = \varepsilon_0 V_g/eD^2$. The classical scaling of $V_g^{3/2}/D^2$ given in (1) immediate follows from $J = env$, with $v = \sqrt{2\,\text{e}V_g/\text{m}}$.

Note that the above derivation has ignored the electron emission mechanisms in the vicinity of the surface. It has also ignored the surface roughness, which may become important in nano-scale diodes.

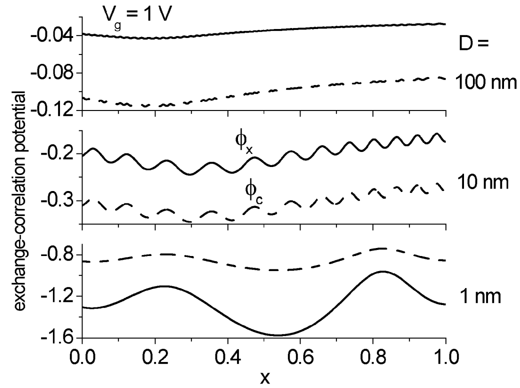

Fig. 4. Calculated normalized exchange-correlation potential for $D = 1, 10$, and 100 nm at $V_g = 1$ V: exchange (solid) and correlation (dashed).

III. Effects of Exchange Correlation

It is important to note that even though the quantum scaling can be derived without the consideration of electron exchange correlation, the electron exchange correlation is important in calculating the quantum enhancement for low-gap voltage and low-gap spacing [12]. These could well be the regimes of practical interest. For example, at $D = 100$ nm and $V_g = 10$ V, the quantum enhancement is, respectively, $\mu_Q = 1.62$ and $\mu_Q = 1.07$ for calculations with and without including the electron exchange correlation. The difference is much greater at lower values of D and V_g. For instance, $\mu_Q = 7.62$ and $\mu_Q = 1.35$ at $D = 10$ nm and $V_g = 1$ V (at the same average electric field of 0.1 V/nm).

The importance of the electron exchange correlation can also be seen from the calculated normalized electron wave amplitude, normalized electric potential, and normalized exchange-correlation potential (< 0) shown in Figs. 2–4 for $D = 1, 10$, and 100 nm at $V_g = 1$ V and zero electron emission energy. In Figs. 2 and 3, the normalized electron wave amplitude q and normalized electric potential ϕ (in terms of V_g) are plotted as a function of normalized x (in terms of D). The figures show that q and ϕ vary significantly between the cases with (solid) and without (dashed) exchange-correlation effects for $D = 10$ and 100 nm. For comparison, the classical limit of $q = \sqrt{2/3}\text{x}^{-1/3}$ is also plotted in Fig. 2 (short dashed line), and the corresponding classical limit for ϕ is $\phi = x^{4/3}$ (not shown).

In Fig. 3, we also see that the electron tunneling is very small when the electron exchange correlation is ignored (dashed lines), and the space charge potential barrier width is only about 0.08 and 0.015 of the gap spacing for $D = 10$ nm and $D = 100$ nm, respectively. They are much smaller than the potential barrier width of 0.65 and 0.375 when the electron exchange correlation is included (solid lines).

In Fig. 4, we plot the variation of the exchange (solid) and correlation (dash) potentials (in terms of V_g) inside the gap, where both of them decrease with increasing of gap spacing. For example, the magnitudes of the exchange and correlation potentials drop from about 0.8–1.6 (at $D = 1$ nm) to about 0.2–0.3 (at $D = 10$ nm), and to about 0.04–0.12 (at $D = 100$ nm).

IV. Conclusion

We have used a dimensional argument to construct the new scaling for the 1-D CL law in the quantum regime. The simple derivation explains the same scaling which was displayed in the numerical solutions [10], [12]. The importance of correlation and exchange is emphasized.

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