Analytical re-derivation of space charge limited current in solids using capacitor model

Y. B. Zhu[1] and L. K. Ang[1,2,a)]
[1]School of Electrical and Electronic Engineering, Nanyang Technological University, Singapore 639798
[2]Singapore University of Technology and Design, Singapore 138682

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In this paper, we have used a capacitor model to reproduce the known analytical formulas of space charge limited current transport inside both trap-free and trap-filled solids in planar geometry. It is found that the approach is simple when compared to the traditional method as the latter involves solving second order differential equation. Exact analytical results can also be obtained for cylindrical diode with an outer radius much larger than the inner radius. © *2011 American Institute of Physics*. [doi:10.1063/1.3658811]

I. INTRODUCTION

For high current transport in a medium, the current is generally termed as space charge limited (SCL) current for which the space charge effects of the injected current are important. For a one-dimensional (1D) vacuum planar gap with spacing D and voltage V_g, the SCL current density is known as the 1D Child-Langmuir (CL) law[1,2] given by

$$J_{CL} = \frac{4\varepsilon_0}{9}\sqrt{\frac{2e}{m}}\frac{V_g^{3/2}}{D^2}, \tag{1}$$

where e and m are the charge and mass of the electron, respectively, and ε_0 is the permittivity of free space. New developments of the CL law include multi-dimensional classical models,[3–5] quantum models,[6–8] short-pulse models,[9–11] and two-dimensional (2D) electromagnetic models.[12]

For a 1D trap-free solid, the corresponding SCL current density is known as the Mott–Gurney (MG) law[13] given by

$$J_{MG} = \frac{9}{8}\varepsilon_0\varepsilon_r\mu\frac{V_g^2}{D^3}, \tag{2}$$

where ε_r is the relative permittivity of the solid, and μ is the electron mobility. If the solid has an exponentially distributed traps (in energy), it is known as the Mark–Helfrich (MH) law,[13,14] or the trap-limited SCL current density J_{MH}

$$J_{MH} = N_c\mu e^{1-l}\left[\frac{\varepsilon_0\varepsilon_r l}{N_t(l+1)}\right]^l\left(\frac{2l+1}{l+1}\right)^{l+1}\frac{V_g^{(l+1)}}{D^{(2l+1)}}. \tag{3}$$

Here, N_c is the effective density of states corresponding to the energy at the bottom of the conduction band, N_t is the total trapped electron density, and $l = T_t/T$ is the ratio of distribution of traps to the free carriers.

There are renew interests in SCL conduction found in many novel devices, such as graphene oxide sheets,[15] light emitting diodes,[16] GaN nanorod,[17] organic device,[18–21] polymer transistor,[22] nanowire,[23] magnetoresistance,[24,25] photocurrent,[26] and nanocrystallites embedded silicon Schottky junction.[27] New models of MG law have also been constructed to study the effect of finite emission area[28] and the presence of a free space between a solid and electrode.[29]

The traditional approach in the formulation of the SCL current density in both free space and solids is by solving the Poisson equation, continuity equation, and equation of motion with the related boundary conditions at the cathode ($z = 0$) and anode ($z = D$): $V(z = 0) = 0$, and $V(z = D) = V_g$, and $dV/dz[z = 0] = 0$ (electric field is zero at the cathode).

It is shown recently that the formulation of the 1D CL law can be reproduced by using a simple capacitor model at steady-state condition[30] and single-electron limit.[31] In this paper, we are interested to show that the SCL current transport in solids, namely the 1D analytical MG and MH law, can be reproduced by using a capacitor model for both planar and cylindrical diodes.

II. TRAP-FREE SOLID CASE

First, consider a trap-free solid diode as a parallel plate capacitor with a total bound charge of Q_b given by

$$Q_b = CV_g, \tag{4}$$

where $C = \varepsilon_0\varepsilon_r A/D$ is the capacitance, A is the area of the plate, and D is the separation of the capacitor. Without the space charge effects, the electric field inside the solid between the plates is $E = V_g/D$. Assuming all the negative bound charges (electrons) are liberated from the cathode at zero-initial velocity with a transit time (across the trap-free solid) of

$$\tau = \frac{D}{v} = \frac{D^2}{\mu V_g}. \tag{5}$$

Here, we have assumed that the velocity of the electrons is determined by the mobility of the solid, which is $v = \mu \times E = \mu V_g/D$. From Eqs. (4) and (5), the electron current density J is simply given by

$$J = \frac{Q_b}{\tau A} = \mu\varepsilon_0\varepsilon_r\frac{V_g^2}{D^3}, \tag{6}$$

a)Electronic mail: elkang@ntu.edu.sg.

which is same as the 1D MG law [see Eq. (2)] except for the numerical constant 9/8. Note this numerical constant can be obtained by using the current continuity equation and the Poisson equation (see below).

From the 1D steady-state continuity equation, $dJ/dz = 0$, we know that J is independent of z, and Eq. (6) implies that the electrical potential field has a scaling of

$$\phi(z) = \frac{V_g}{D^{3/2}} z^{3/2}. \tag{7}$$

By inserting Eq. (7) into the Poisson equation, we obtain the electron charge density ρ, which is

$$\rho(z) = \varepsilon_0 \varepsilon_r \phi''(z) = \varepsilon_0 \varepsilon_r \frac{3}{4} \frac{V_g}{D^{3/2}} z^{-1/2}. \tag{8}$$

Finally, the electron current density can be determined by evaluating the velocity and the electron charge density collected at the anode ($z = D$), which are $v(D) = \frac{3}{2}\mu V_g/D$, and $\rho(D) = \frac{3}{4}\varepsilon_0\varepsilon_r\varepsilon_g/D^2$, respectively.

In doing so, we recover the 1D SCL current density for a trap-free solid or the 1D MG law as shown in Eq. (2) given by

$$J = \rho(D)v(D) = \frac{9}{8}\mu\varepsilon_0\varepsilon_r \frac{V_g^2}{D^3}. \tag{9}$$

Note in this method of formulation, there is no need to solve the Poisson equation with the boundary conditions (as done in the traditional approach), except a simple differentiation as shown in Eq. (8).

III. TRAP-FILLED SOLID CASE

Similarly, the same approach can be used to recover the 1D MH law for a trap-filled solid with an exponential trap distribution written as

$$N(E) = N_0 \exp[(E - E_c)/kT_t], \tag{10}$$

where $N_0 = N_t/kT_t$, N_t is the total trapped electron density, T_t is the temperature of the trap distribution, E_c is the energy at the bottom of the conduction band, and k is the Boltzmann constant. For a constant applied electric field $E = V_g/D$, we assume that the quasi Fermi level across the sample $E_f(z)$ is also constant defined as E_f. Under this condition, the density of the trapped electrons is

$$n_t = \rho_t/e = N_t \exp\left[\left(E_f - E_c\right)/kT_t\right]. \tag{11}$$

Similarly, the density of the free electrons at the valence band is

$$n_f = \rho_f/e = N_c \exp\left[\left(E_f - E_c\right)/kT\right], \tag{12}$$

where N_c is the effective density of states at the valence band. By eliminating E_f from Eqs. (11) and (12), we obtain the relationship between n_f and n_t as

$$n_f = N_c \left(\frac{n_t}{N_t}\right)^l. \tag{13}$$

Consider the injected electrons are from the bound charge on the capacitor $Q_b = CV_g$, which will fill the traps giving an expression of n_t

$$n_t = \frac{CV_g}{eDA} = \frac{\varepsilon_0\varepsilon_r V_g}{eD^2}. \tag{14}$$

By combining Eqs. (13) and (14), the charge density of the free electrons is

$$\rho_f = eN_c \left(\frac{\varepsilon_0\varepsilon_r V_g}{eN_t D^2}\right)^l. \tag{15}$$

By knowing the velocity of the electrons is $v = \mu V_g/D$, the electron current density $J = \rho_f \times v$ becomes

$$J = e\mu N_c \left(\frac{\varepsilon_0\varepsilon_r}{eN_t}\right)^l \frac{V_g^{l+1}}{D^{2l+1}}. \tag{16}$$

From the continuity equation (J is a constant), Eq. (16) shows a scaling of

$$J \propto \frac{\phi^{l+1}(z)}{z^{2l+1}} = \frac{V_g^{l+1}}{D^{2l+1}}, \tag{17}$$

which implies that the electrical potential is in the form of

$$\phi(z) = \frac{V_g}{D^{(2l+1)/(l+1)}} z^{(2l+1)/(l+1)}. \tag{18}$$

From Eq. (18) and the Poisson's equation, we can then obtain the trapped charge density as

$$\rho_t(z) = \varepsilon_0\varepsilon_r \frac{l(2l+1)}{(l+1)^2} \frac{V_g}{D^{(2l+1)/(l+1)}} z^{-1/(l+1)}, \tag{19}$$

and thus gives free electron density as [from Eq. (13)]

$$\rho_f(z) = eN_c \left[\frac{\varepsilon_0\varepsilon_r(2l+1)l}{eN_t(l+1)^2} \frac{V_g}{D^{(2l+1)/(l+1)}} z^{-1/(l+1)}\right]^l. \tag{20}$$

Finally, by knowing the drift velocity at anode ($z = D$) is $v(D) = \mu\frac{2l+1}{l+1}\frac{V_g}{D}$, and by determining the free charge density [Eq. (20) with $z = D$], we recover the space charge limited current for a trap-filled solid or [the 1D MH law as shown in Eq. (3)] as $J_{MH} = \rho(D)v(D)$

$$J_{MH} = N_c^{(1-l)} \left[\frac{\varepsilon_0\varepsilon_r l}{N_t(l+1)}\right]^l \left(\frac{2l+1}{l+1}\right)^{(l+1)} \frac{V_g^{(l+1)}}{D^{(2l+1)}}. \tag{21}$$

IV. CYLINDRICAL DIODE

The 1D MG and MH law for a planar diode has also been extended to a cylindrical model,[32–34] where the analytical formula has been obtained at the limit of very small cathode's radius when compared to the anode's radius ($r_c \ll r_a$). In this section, we will use the same capacitor model to reproduce the analytical formulas for both trap-free and trap-filled solids (see below).

For a cylindrical diode, the capacitance (per unit length) is

$$C = \frac{2\pi\epsilon_r\epsilon_0}{\ln(r_a/r_c)}. \quad (22)$$

The mean velocity for an electron to transport across the cylindrical gap is

$$\langle v\rangle = \mu\frac{Q}{2\pi\epsilon_r\epsilon_0}\frac{\ln(r_a/r_c)}{r_a - r_c}, \quad (23)$$

with a transit time of $\langle\tau\rangle = (r_a - r_c)/\langle v\rangle$. The total current line density I [A/m] $= Q/\langle\tau\rangle$ is

$$I = \frac{Q}{r_a - r_c}\langle v\rangle = 2\pi\epsilon_r\epsilon_0\mu\frac{V_g^2}{\ln(r_a/r_c)(r_a - r_c)^2}. \quad (24)$$

In the limit of $r_a \gg r_c$, we can treat the $\ln(r_a/r_c)$ term as a constant to obtain a scaling of current line density $I \propto V_g^2/r_a^2$, which implies that the electrical potential is in the form of

$$\phi(r) = \frac{V_g}{r_a}r. \quad (25)$$

Finally, by evaluating the electron's velocity and density at $r = r_a \gg r_c$, as

$$v(r_a) = \mu\left|\frac{d\phi(r)}{dr}\right|_{r=r_a} = \mu\frac{V_g}{r_a}, \quad (26)$$

$$\rho(r_a) = \epsilon_0\epsilon_r\frac{1}{r}\frac{d}{dr}\left[r\frac{d\phi(r)}{dr}\right]_{r=r_a} = \epsilon_0\epsilon_r\frac{V_g}{r_a^2}, \quad (27)$$

we recover the space charge limited current line density for a cylindrical trap-free diode[32]

$$I = 2\pi r_a\rho(r_a)v(r_a) = 2\pi\epsilon_0\epsilon_r\mu\frac{V_g^2}{r_a^2}. \quad (28)$$

To extend the cylindrical model for a trap-filled solid, we first suppose the injected electrons will fill the traps in a capacitor model, given by

$$Q_t = \pi\left(r_a^2 - r_c^2\right)\rho_t = CV_g. \quad (29)$$

From Eq. (13), the line charge density of the free electrons is

$$Q_f = \pi\left(r_a^2 - r_c^2\right)N_c e\left[\frac{CV_g}{\pi\left(r_a^2 - r_c^2\right)eN_t}\right]^l. \quad (30)$$

From Eq. (23), we may solve for the mean velocity $\langle v\rangle$ with $Q = Q_t$ [from Eq. (29)], and calculate the current line density as

$$I = \frac{Q_f}{r_a - r_c}\langle v\rangle \propto \frac{V_g^{l+1}}{(r_a - r_c)^2\ln\left(\frac{r_a}{r_c}\right)^l\left(r_a^2 - r_c^2\right)^{l-1}}. \quad (31)$$

In the limit of $r_a \gg r_c$, Eq. (31) becomes

$$I \propto \frac{V_g^{l+1}}{r_a^{2l}} = \frac{\phi(r)^{l+1}}{r^{2l}}, \quad (32)$$

which implies that potential distribution is the form of

$$\phi(r) = \frac{V_g}{r_a^{2l/(l+1)}}r^{2l/(l+1)}. \quad (33)$$

By using the electron mobility equation, the Poisson equation, and Eq. (13), we can calculate the electron's velocity, the trap charge density, and free electron density (all at $r = r_a \gg r_c$), respectively, as

$$v(r_a) = \mu\frac{V_g}{r_a}\frac{2l}{l+1}, \quad (34)$$

$$\rho_t(r_a) = \epsilon_0\epsilon_r\left(\frac{2l}{l+1}\right)^2\frac{V_g}{r_a^2}, \quad (35)$$

$$\rho_f(r_a) = eN_c\left[\frac{\epsilon_0\epsilon_r}{eN_t}\left(\frac{2l}{l+1}\right)^2\frac{V_g}{r_a^2}\right]^l. \quad (36)$$

Finally, using Eqs. (34) and (36), we recover the space charge limited current line density for a cylindrical trap-filled diode[32]

$$I = 2\pi eN_c\mu\left[\frac{\epsilon_0\epsilon_r}{eN_t}\right]^l\left(\frac{2l}{l+1}\right)^{2l+1}\frac{V_g^{l+1}}{r_a^{2l}}. \quad (37)$$

V. CONCLUSION

In summary, we have confirmed that the capacitor approach[30] that was successful in the re-derivation of the SCL current in free space or the 1D CL law is also applicable for the SCL conduction in both trap-free and trap-filled solids, which are known as the 1D MG law and 1D MH law, respectively. It will be of interests for further investigations to test the capacitor model to other SCL models such as SCL bipolar (electron and hole) conduction in organic electronics, cylindrical CL law, 2D models, and also to different physical regimes including relativistic and quantum effects, just to name a few. It is noted that this effort might provide insight for the material-by-design community by clearly showing what physical effects are important in the engineering of densities of states in novel materials and geometries.

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