Two-dimensional space-charge-limited flows in a crossed-field gap

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This letter presents a two-dimensional (2D) model of space-charge-limited current in a planar crossed-field gap with a magnetic strength of $B/B_H=0-3$, where B_H is the Hull cutoff magnetic field. The electrons are emitted from an infinite length strip of finite width W comparable to the gap spacing D. It is found that the 2D enhancement of the crossed-field limiting current is $1+F\times 4D/(\pi W)$, where F (=0.05–0.5) is a normalized mean-position factor, and it is a function of B/B_H. Good agreement has been obtained in comparisons with particle-in-cell simulation. © *2007 American Institute of Physics*. [DOI: 10.1063/1.2720710]

For high current electron beam transport across an anode-cathode (AK) vacuum gap, the space charge effects of the electrons will limit the amount of current propagation. In the one-dimensional (1D) nonrelativistic case, the maximum steady-state current density that can be transported across a planar gap is given by the classical Child-Langmuir (CL) law,[1]

$$J_{\mathrm{CL}}=\frac{4\epsilon_0}{9}\sqrt{\frac{2e}{m_e}}\frac{V_g^{3/2}}{D^2}, \tag{1}$$

where D is the gap spacing, V_g is the gap voltage, ϵ_0 is the free space permittivity, and e and m_e are the charge and mass of the electron, respectively. Recently, the 1D classical CL law has been extended to multidimensional classical models,[2–4] a two-dimensional (2D) weakly relativistic model,[4] and quasi-2D quantum model.[5,6] Similar 2D models have been also formulated for the short-pulse CL law[7] and the cylindrical configuration.[4]

For a 1D crossed-field gap with a magnetic field B parallel to the electrode surface, limiting current density (due to space charge effects) has been formulated for $B \leqslant B_H$[8] and $B \geqslant B_H$,[9] where $B_H=\sqrt{2m_eV_g/eD^2}$ is the Hull cutoff magnetic field that prevents electrons (without space charge effects) from reaching the anode.[10] The classical 1D model was also extended to a quantum model for a nanosized crossed-field gap.[11] For the $B \geqslant B_H$ case, it was found that transition from laminar flow to turbulent flow may occur due to over injection of current with a value higher than the limiting current value,[9] small ac voltage modulation,[12] resistivity,[13] and misalignment of the magnetic field.[14]

The microwave magnetron is a type of crossed-field device that is widely used in microwave ovens, radars, industrial heating, medical accelerators, and plasma sources. Recently, low noise and fast mode locking for the microwave magnetron have been developed using various techniques such as magnetic priming,[15] cathode priming,[16] multiple cathodes of narrow width,[17] and transparent cathode.[18] For the methods using small-sized cathode,[17,18] the size of the cathode can be comparable to the spacing between the AK gap for which the 1D models[8,9] determine that the limiting current are no longer valid. A cooling method of using electron field-thermal emission in a crossed-field gap has also been proposed, which applies a sharp emitter with a size much smaller than the gap spacing.[19]

Thus, it is of interest to study the effects of finite emission area on the space-charge-limited (SCL) electron flows in a crossed-field gap. For simplicity, we will consider a simple planar gap with electron emission from an infinitely long strip with a width comparable to the gap spacing.

Consider a 1D planar gap of spacing D with a grounded cathode and an anode potential of V_g. Electrons with zero emission energy ($u_0=0$) are injected normally into the gap from the cathode at $x=0$ to the anode at $x=D$. The electron emission is restricted in an infinitely long strip (along the z direction) with a finite width of W from $y=-W/2$ to $y=W/2$. The transverse magnetic field (either B_y or B_z) is applied parallel to the electrode surfaces. By using the previous methods,[3,4] the enhancement of the 2D SCL current density in terms of the 1D SCL current density can be expressed as $\mu_{\mathrm{cf}}=J_{\mathrm{cf}}(2\mathrm{D})/J_{\mathrm{cf}}(1\mathrm{D})=1+F\times G$, which is

$$\frac{J_{\mathrm{cf}}(2\mathrm{D})}{J_{\mathrm{cf}}(1\mathrm{D})}=1+\frac{4}{\pi}\frac{F}{W/D}. \tag{2}$$

Here, $F=\int_0^D(x/D)n(x)dx/\int_0^D n(x)dx$ is the normalized mean-position factor of the 1D SCL flow at various B/B_H (see below) and $G=4\mathrm{D}/(\pi W)$ is the geometrical factor independent of B/B_H.[4] Note that the equation is only valid for $W/D \geqslant 1$.

In the formulation of F as a function of B/B_H, we follow the conventions used in the previous papers,[8,9] with the following normalized variables: $\Omega=eB/m$ is the normalized magnetic field strength, $\tilde{x}=x/D$ is the normalized position in the gap, $\tilde{t}=\Omega t$ is the normalized transit time, $\tilde{u}=u/\Omega D$ is the normalized velocity, $\tilde{V}=eV/m\Omega^2D$ is the normalized potential in the gap, $\tilde{c}=c/\Omega^2D$ is the normalized initial acceleration, $\tilde{E}=eE/m\Omega^2D$ is the normalized electric field, and $\tilde{J}=eJ/m\epsilon_0\Omega^3D$ is the normalized current density. By changing the variable to $t=\int dx/u$, the normalized mean-position correction factor F can be expressed as

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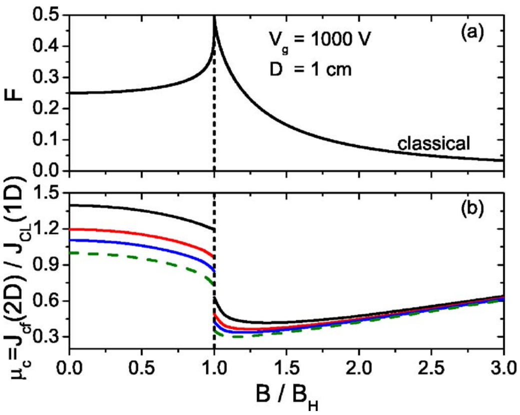

FIG. 1. (Color online) (a)Mean-position correction factor F and (b) Normalized 2D current density (in terms of the 1D CL law) $\mu_c = J_{cf}(2D)/J_{CL}(1D)$ as a function of B/B_H for W=0.8, 1.6 and 3 (solid lines: top to bottom). The dashed line denotes the normalized 1D results $\mu_c = J_{cf}(1D)/J_{CL}(1D)$ obtained from Refs. 8 and 9. Zero initial injection energy is assumed in the analytical model.

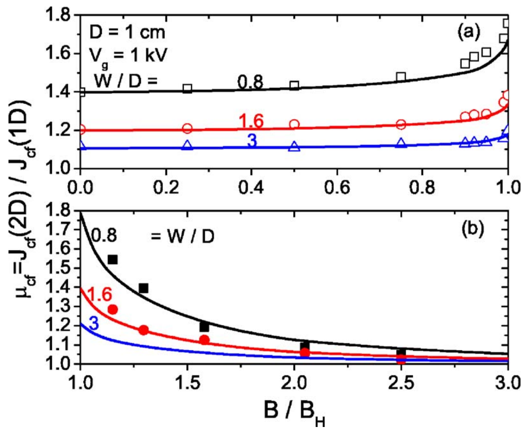

FIG. 2. (Color online) Normalized 2D current density (in terms of the 1D crossed-field CL law) $\mu_{cf} = J_{cf}(2D)/J_{cf}(1D)$ for W=0.8, 1.6, and 3 (solid lines) as functions of (a) $B/B_H \leqslant 1$, and (b) $B/B_H > 1$. The unfilled symbols denote the simulation results obtained from XOOPIC for $B/B_H \leqslant 1$ with $B = B_z$. The filled symbols denote the simulation results obtained from XOOPIC for $B/B_H > 1$ with $B = B_y$. Initial injection energy of 0.01 eV is used in the simulation.

$$F = \frac{1}{\tilde{T}}\int_0^{\tilde{T}} \tilde{x}(\tilde{t})d\tilde{t} = \frac{1}{\tilde{T}}[m\tilde{J}[\tilde{T}^2/2 + \cos\tilde{T} - 1] + \tilde{c}[\tilde{T} - \sin\tilde{T}]]. \tag{3}$$

For $B/B_H < 1$ case, we have $\tilde{c} = (\sqrt{(B_H/B)^2 - 1} - \tilde{J} + \tilde{J}\cos\tilde{T})/\sin\tilde{T}$ and m=1. For $B/B_H > 1$ case, they are $\tilde{c} = -2\tilde{J}\tan(\tilde{T}/2)$ and m=2. Note that the m=2 is to account for the return injected current back to the cathode when $B > B_H$. Here, $\tilde{J}$ and $\tilde{T}$ are solved as a function of B/B_H by using the 1D SCL electron flow model in a crossed-field gap. The methodology of solving them can be found elsewhere,[8,9] and will not be elaborated here. At B=0, we recover the limit of zero magnetic field, which gives F=0.25.[3,4]

Figure 1(a) illustrates the variation of the mean-position factor F as a function of B/B_H. At the transition point of B/B_H=1, we have F=0.5. The figure indicates that F increases with $B/B_H \to 1$ and decreases with $B/B_H > 1$. In Fig. 1(b), the 2D crossed-field limiting current density $J_{cf}(2D)$ [normalized with respect to the 1D CL law $J_{CL}(1D)$] is plotted as a function of B/B_H=0–3 to illustrate the enhancement of the SCL current due to finite values of W/D=0.8, 1.6, and 3 (solid lines). The 1D results of $W/D \gg 1$ (dashed lines) are also plotted for comparison. From the figure, we see that the 2D enhancement of SCL flow at $B/B_H > 1$ is less important as compared to $B/B_H < 1$. The 2D effects become insignificant at large magnetic field due to small Brillouin hub height x_T as compared to gap spacing and the width of the emission strip ($x_T < D \leqslant W$).

Our calculated results have been verified by using a 2D particle-in-cell (PIC) code called XOOPIC[20] in the range of $0 \leqslant B/B_H \leqslant 3$. The simulation model in XOOPIC consists of a planar gap of $x = D$=1 cm and V_g=1 kV. Electrons are emitted uniformly from a 2D infinite strip of width W (in y direction) with a small initial energy of 0.01 eV to initiate the emission. For $B/B_H < 1$, we applied a static transverse magnetic field in the z direction, $B = B_z$. By using the over injection method, we determine the SCL current by gradually increasing the emission current until a limiting value for which virtual cathode oscillation will occur. The normalized 2D crossed-field SCL current density $\mu_{cf} = J_{cf}(2D)/J_{cf}(1D)$ is plotted from B/B_H=0 to 1 (solid lines) for W=0.8, 1.6, and 3, as shown in Fig. 2(a). The symbols in the figure represent the XOOPIC simulation results obtained at various W/D. From Fig. 2(a), it is obvious that we have excellent agreements between the calculated and the simulated results. The error increases as B/B_H approaches unity and is only 5% for the case of W=0.8 cm for B/B_H very close to 1. The discrepancies between the calculated and XOOPIC results are much smaller (i.e., up to 3.2% for W/D=1.6 and up to 1.3% for W/D=3). For $W \gg D$, we will recover the 1D crossed-field SCL current density[8] (not shown).

For $B/B_H > 1$ case, a static transverse magnetic field in the y direction, $B = B_y$, is used in the PIC simulation. In this case, the limiting current density is determined by observing the onset of turbulent flow (or collapse of laminar flow) when the injected current is higher than a critical value at various W/D for $B/B_H > 1$ case. The onset of the turbulent flow can be determined by looking at the phase-space plots and the electron density distribution along the center of the emission width obtained from XOOPIC (see below). In Fig. 2(b), XOOPIC simulations are performed at W/D=0.8 and 1.6 cases with errors of 6% and 4.5%, respectively. For larger W/D, the error is expected to be even smaller. In Fig. 3, the phase-space plots $u(x)$ (in x direction) are shown for the injected current at B/B_H=2.05 and W/D=1.6 when the injected current is (a) 5% lower and (b) 5% higher than the 2D limiting current density. The dashed lines (red) in the figure illustrate the transition of the laminar flow shown in Fig. 3(b) to the onset of turbulent flow shown in Fig. 3(a), when the injected current is increased to a value about the 2D limiting current density.

It is important to note that our analytical model is only valid with the transverse magnetic field along the y direction ($B = B_y$) for $B > B_H$ case. This is to ensure that the return electrons (from Brillouin hub position) will be back to the emission strip to be consistent with the 1D model.[9] If a B_z field is used, the return electrons do not interact with the injected electrons especially for small W/D, and thus the analytical model becomes invalid.

In conclusion, we have extended the previous 2D classical models of SCL electron flows in a normal AK gap[2–4] to a

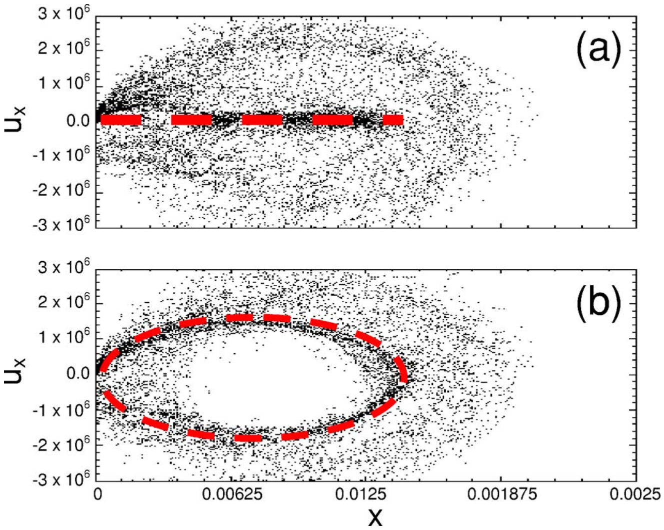

FIG. 3. (Color online) Velocity phase-space plot u_x (in x direction) at $B/B_H=2.05$ and $W/D=1.6$ with an injected current of (a) 5% lower and (b) 5% higher than the 2D limiting current density. Initial injection energy of 0.01 eV is used in the simulation

crossed-field gap for both $B<B_H$ and $B>B_H$ cases. Our results indicate that the 2D enhancement of limiting current density is more significant for $B<B_H$ case. Our analytical results showed excellent agreements with XOOPIC simulations for a wide range of B/B_H and $W/D\geq 1$. It is worth mentioning that the 2D model developed here is readily to be extended to three-dimensional (3D) models with more complicated emission shapes (such as rectangle and ellipse) by simply changing the G parameter to various formulas that have been derived before [see Eqs. (10) and (11) in Ref. 4]. The verification of such 3D models will require intensive 3D PIC simulation, which is beyond the scope of this letter.

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