Modern Physics Letters A
Vol. 31, No. 29 (2016) 1650168 (7 pages)

DOI: 10.1142/S0217732316501686

World Scientific
www.worldscientific.com

Zitterbewegung near a Schwarzschild-type black hole

Alexander V. Zhukov*,§, Roland Bouffanais*,
Natalia N. Konobeeva† and Mikhail B. Belonenko†,‡

*Singapore University of Technology and Design, 8 Somapah Road, Singapore 487372
†Volgograd State University, 400062 Volgograd, Russia
‡Laboratory of Nanotechnology, Volgograd Institute of Business, 400048 Volgograd, Russia
§alex_zhukov@sutd.edu.sg

Received 9 June 2016
Accepted 19 July 2016
Published 25 August 2016

In this paper, we present an approximate analytic solution to the Dirac equation in the Schwarzschild geometry to investigate the Zitterbewegung (ZB) effect. The analytical expression for the current density, which is induced by the motion of a wave packet of electrons, is obtained. In addition, the intensity of dipole radiation near the black hole is calculated. The proposed approach is based on the Schrödinger representation, thereby allowing to consider the ZB effect in the case of a curved space. The considered example demonstrates a possibility for applying this approach to astrophysical applications, in particular to problems of the electron radiation in the vicinity of real black holes.

Keywords: Zitterbewegung; Dirac equation; dipole radiation; black hole.

PACS Nos.: 03.65.Pm, 03.65.Ta, 04.70.-s

1. Introduction

In the last few years, there has been a growing interest in the study of black holes, especially in relation to possible applications of the physics of black holes aspects to the solid state and plasma physics.[1–4] It should be noted that a more detailed study of the Dirac equation properties near black holes allows us to provide data on the characteristics of the Hawking radiation and other quantum effects.[5,6] One of the most striking features of quantum electrodynamics is the Zitterbewegung (ZB) effect. Note that the problem of ZB observation became of special interest for applications after the discovery of graphene, which is described in the long-wave approximation by the Dirac equation.[7] Recent research has demonstrated some progress in the experimental study of the ZB.[8] Nevertheless, there is still a wide range of interesting and relevant problems to be addressed. In particular, the features of this effect in curved space bear significant importance for a body of astrophysical applications. In Ref. 9, an approach based on the Schrödinger representation has been proposed, which allowed to consider the effect of ZB in the case of curved space for a classical "toy" model of a black hole. We choose this model because the corresponding system of functions is known in analytically in a closed form. Obviously, any discussion about possible real-world applications should consider Schwarzschild-like black holes, whose existence has been proved. For this reason, the present work presents the theory of the ZB effect in curved space for the spherically symmetric Schwarzschild black hole.

2. Fundamental Equations

Using spherical coordinates, the metric for the Schwarzschild black hole has the following classical form:[10]

$$ds^2 = c^2\,dt^2 - \frac{dr^2}{u(r)^2} - \frac{r^2}{v(r)^2}(d\theta^2 + \sin^2\theta\,d\varphi^2)\,, \tag{1}$$

where ds is the line element, c the speed of light ((hereinafter to be equal to unity),

$$u(r) = 1 - \frac{r_0}{r}\,,$$

$$v(r) = \sqrt{1 - \frac{r_0}{r}}\,,$$

with r_0 the Schwarzschild radius of the massive body. Properties of fermions near the black hole are described on the basis of the Dirac equation generalized for the case of a curved spacetime:[11]

$$\gamma^\mu(\partial_\mu - \Omega_\mu)\psi = 0\,. \tag{2}$$

Here and thereafter, the repeated indices are implicitly summed over; ∂_μ is the partial derivative with respect to coordinate μ, Ω_μ is the component of the spin connection, ψ is the wave function (column vector). According to Ref. 12, if one is given the following metric tensor

$$\begin{aligned} ds^2 &= g_{\alpha\beta}\,dx^\alpha\,dx^\beta\,,\\ g_{\alpha\beta}g^{\beta\gamma} &= \delta^\gamma_\alpha\,, \end{aligned} \tag{3}$$

(δ^γ_α being the Kronecker delta) then one can define the following tetrad $\{e_\alpha\}$ via the metric tensor g and the Lorentz metric η:

$$\begin{aligned} g_{\alpha\beta} &= e^a_\alpha e^b_\beta \eta_{ab}\,,\\ g^{\alpha\beta} &= e^\alpha_a e^\beta_b \eta^{ab}\,,\\ \eta_{ab}\eta^{bc} &= \delta^c_a\,. \end{aligned} \tag{4}$$

Then,

$$\Omega_\mu = \frac{1}{4}\gamma_a\gamma_b e^a_\lambda g^{\lambda\sigma}(\partial_\mu e^b_\sigma - \Gamma^\lambda_{\mu\sigma}e^b_\lambda)\,,$$

$$\Gamma^\lambda_{\mu\sigma} = \frac{1}{2}g^{\lambda\nu}(g_{\sigma\nu,\mu} + g_{\nu\mu,\sigma} - g_{\mu\sigma,\nu})\,,$$

$$\gamma^\mu = e^\mu_a\gamma_a\,.$$

For the metrics (1), the only nonzero Christoffel symbols are

$$\Gamma^1_{11} = -u\dot{u}, \qquad \Gamma^1_{22} = -u^2\left(\frac{vr - r^2\dot{v}}{v^3}\right), \qquad \Gamma^1_{33} = -u^2\sin^2\theta\left(\frac{vr - r^2\dot{v}}{v^3}\right),$$

$$\Gamma^2_{12} = \frac{vr - r^2\dot{v}}{rv}\,, \qquad \Gamma^3_{23} = \cot\theta\,, \qquad \Gamma^3_{13} = \frac{v - r\dot{v}}{rv}\,.$$

The spin connections are then given by

$$\Omega_0 = \Omega_1 = 0\,, \qquad \Omega_2 = \gamma_1\gamma_2 u\frac{v - r\dot{v}}{2v^2}\,, \qquad \Omega_3 = \gamma_1\gamma_3 u\frac{v - r\dot{v}}{2v^2} + \frac{1}{4}\gamma_2\gamma_3\cos\theta\,.$$

From the Dirac equation (2), we obtain the Hamiltonian in the form

$$H = \begin{pmatrix} mv(r) & -u(r)\partial_r + k\dfrac{v(r)}{r} \\ u(r)\partial_r + k\dfrac{v(r)}{r} & -mv(r) \end{pmatrix},$$

where m is the fermion mass, k is the angle quantum number, which is determined by

$$k = \text{either} \quad j + 0.5 = l(j = l - 0.5) \quad \text{or} \quad j + 0.5 = l + 1(j - 0.5 = l)\,.$$

The equation of motion can therefore be cast as[13]

$$\dot{\hat{r}} = -i[\hat{H}, \hat{r}]\,. \tag{5}$$

Note that the standard approach to ZB (based on Heisenberg's framework) yields some problems. It is easy to see that solving the ZB problem in a curved spacetime is rather difficult. This is due to the nonclosed system of operator equations in the curved space. To overcome this, one can resort to dealing with this problem using Schrödinger's theoretical framework instead:

$$\dot{\hat{r}} = \hat{r}|_0\,, \qquad |\psi\rangle = |\psi(t)\rangle\,,$$

so that the current density is given by

$$j = \langle\psi(t)|\dot{r}|\psi(t)\rangle\,. \tag{6}$$

Given Schrödinger's equation $i\dot{\psi} = H\psi$, or equivalently its eigenvalue representations $|\psi(t)\rangle = \sum_\alpha C_\alpha \exp(i\epsilon_\alpha t)|\psi(t)\rangle_\alpha|_0$, the current density can be recast as

$$j = -i\langle\psi(t)|[r, H]|\psi(t)\rangle\,, \tag{7}$$

where the commutator is given by

$$[r, H] = \begin{pmatrix} 0 & u(r) \\ -u(r) & 0 \end{pmatrix}.$$

Let us suppose there is a solution $\psi(t) = |\psi_1(t); \psi_2(t)\rangle$, then we have the following expression for the current density:

$$\begin{aligned} j(r,t) = 2C_1C_2 \int d\omega \exp\left(\frac{(\omega-\omega_0)^2}{\Delta^2}\right) \sin(2\omega t) u(r)\{\psi_2^*(\omega)\psi_1(-\omega) \\ + \psi_2^*(-\omega)\psi_1(\omega) + \psi_1^*(\omega)\psi_2(-\omega) + \psi_1^*(-\omega)\psi_2(\omega)\}\,, \end{aligned} \tag{8}$$

where the prefactors C_1 and C_2 are constants, ω_0 is the central frequency in the energy representation and Δ is the packet width. The wave function can be represented as follows:

$$\psi(r,\theta) = \begin{pmatrix} \psi_1(y) \\ \psi_2(y) \end{pmatrix} \Phi(\theta)\,. \tag{9}$$

With all the above, the following general solution is obtained:

$$\psi(r,\theta) = \int d\omega \exp\left(\frac{(\omega-\omega_0)^2}{\Delta^2}\right)\psi(\omega)\Phi(\theta)\,,$$

$$\begin{aligned} \psi_1(\omega) &= C_1^1\omega^{-1}M_{r_+,s}(2i\nu\omega^2) + C_2^1\omega^{-1}W_{r_+,s}(2i\nu\omega^2)\,, \\ \psi_2(\omega) &= C_1^2\omega^{-1}M_{r_-,s}(2i\nu\omega^2) + C_2^2\omega^{-1}W_{r_-,s}(2i\nu\omega^2)\,, \end{aligned} \tag{10}$$

$$s = \sqrt{k^2 + \mu^2 r^2 - \varepsilon^2}\,, \qquad r_\pm = \pm 0.5 - iq\,, \qquad q = \nu + \mu^2(1-\delta)/\nu\,,$$

$$\nu = \sqrt{\varepsilon^2 - \mu^2}\,, \qquad \mu = r_0 m\,, \qquad \varepsilon = r_0 t\,, \qquad r_0 = 2M_{\mathrm{bh}}G\,,$$

$$\begin{aligned} M_{k,\mu}(z) &= \exp(-z/2)z^{\mu+0.5}M(\mu - k + 0.5; 1 + 2\mu; z)\,, \\ W_{k,\mu}(z) &= \exp(-z/2)z^{\mu+0.5}U(\mu - k + 0.5; 1 + 2\mu; z)\,. \end{aligned} \tag{11}$$

Here M_{bh} is the black hole mass, G is the gravitational constant (hereinafter to be equal to unity), C_i^j are the constants, $M(a; b; z)$ and $U(a; b; z)$ are Kummer's functions:[14]

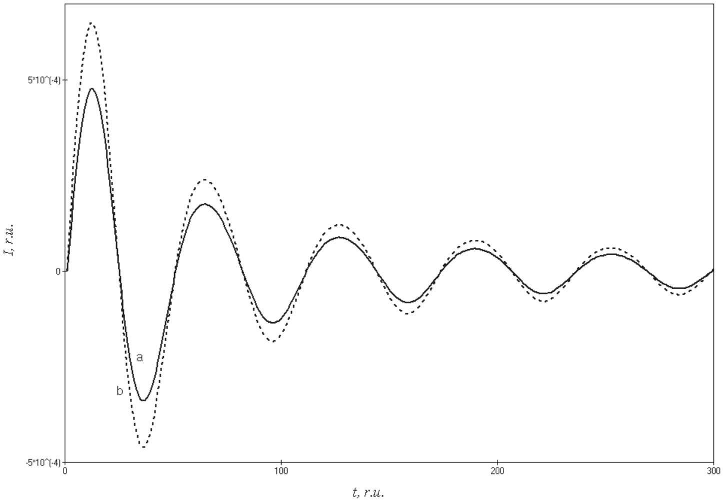

Fig. 1. Dependence of the current density (in relative units) on time ($\Delta = 0.05$): (a) $r = 4$ and (b) $r = 4.4$.

$$M(a;b;z) = \sum_{n=0}^{\infty} \frac{a^{(n)} z^n}{b^{(n)} n!}, \tag{12}$$

$$U(a;b;z) = \frac{\Gamma(1-b)}{\Gamma(a-b+1)} M(a;b;z) + \frac{\Gamma(b-1)}{\Gamma(a)} z^{1-b} M(a-b+1; 2-b; z). \tag{13}$$

It should be noted that a Gaussian form of the wave packet was arbitrarily chosen. However, this particular choice does not affect the generality of the analysis since the developed method makes no specific use of the form of the wave packet.

3. Results and Discussions

First, we present the evolution of the average of current operator for different values of r (see Fig. 1). Note, all the values are given in dimensionless units, given that the mass of black holes may vary within a wide range. Figure 1 shows that one can observe the ZB effect near a Schwarzschild black hole. The influence of the parameter r is obvious, since the effect of "trembling motion" is enhanced with increasing r. This important observation can easily be explained when acknowledging that r corresponds to the radial distribution of the electrons density. With increasing r, the electron density distribution becomes more delocalized.

As a next step, we turn to the dependence of the current density on time for different values of the packet width Δ (see Fig. 2). Note, this result is in agreement with all known results on the ZB effect, since the "damping" value of the trembling

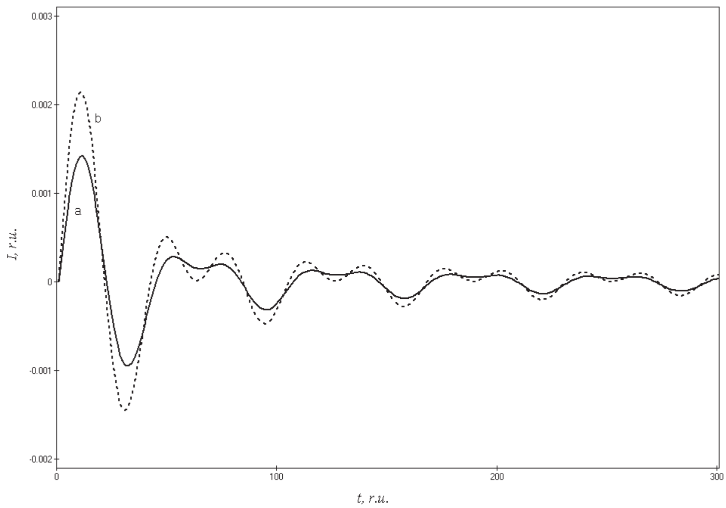

Fig. 2. Dependence of the current density (in relative units) on time ($r = 4$): (a) $\Delta = 0.1$ and (b) $\Delta = 0.5$.

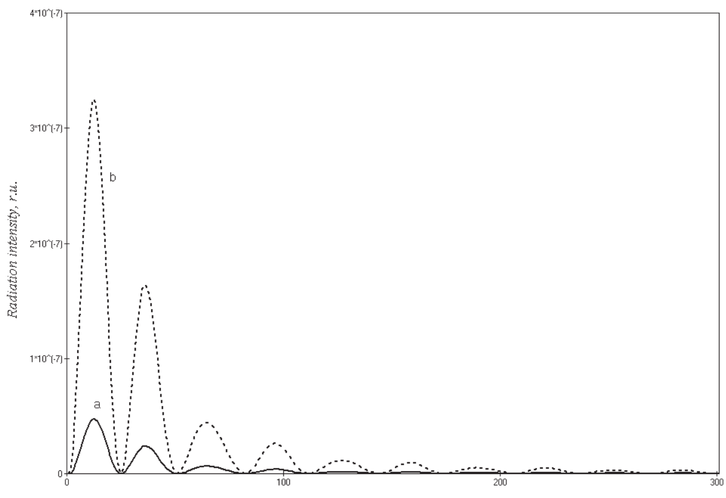

Fig. 3. Dependence of the dipolar radiation intensity on time ($\Delta = 0.05$): (a) $r = 4$ and (b) $r = 4.4$.

motion is directly related to the width of the packet. This so-called "damping" is absent for zero packet width, and the trembling motion continues indefinitely.

Lastly, we study the intensity of the dipole radiation near the black hole. We show that it is proportional to the square of the derivative of the current (see Fig. 3). From this dependence, it can be concluded that the presence of the ZB effect leads to the possible observation of additional peaks in the electromagnetic radiation from the massive charged particles. Those peaks are not caused by accretion or the Hawking effect. They can be useful not only for the experimental observation of the ZB effect, but also for the study of the black hole parameters.

In conclusion, our proposed approach — based on Schrödinger's representation for evolution — allows us to consider the ZB effect in the case of curved spaces. The above example demonstrates the possibility of applying this approach to astrophysical applications, in particular to problems of the electron radiation near the real black holes. This effect can be a reason for extra electron radiation near the black holes and for a change in the emitted spectrum of the electromagnetic waves. This work suggests that the observed effect may induce changes to the spectra. Also the given consideration of the ZB effect can be used in investigations of the neutrino dynamics in the field of the black holes.

Acknowledgments

A.V.Z. and R.B. are financially supported by the SUTD-MIT International Design Centre (IDC). N.N.K. and M.B.B. acknowledge support from the Russian Foundation for Basic Research (project No. 16-07-01265 A).

References

1. D. Chowdhury, S. Raju, S. Sachdev, A. Singh and Ph. Strack, *Phys. Rev. B* **87**, 085138 (2013).
2. C. Charmousis, B. Gouteraux, B. S. Kim, E. Kiritsis and R. Mayer, *JHEP* **1011**, 151 (2010).
3. G. Policastro, D. T. Son and A. O. Starinets, *JHEP* **9**, 43 (2002).
4. Sh. S. Pal, *Phys. Rev. D* **84**, 126009 (2011).
5. H. P. Nollert, *Class. Quantum Grav.* **16**, R159 (1999).
6. S. Chandrasekhar, *The Mathematical Theory of Black Holes* (Oxford Univ. Press, 1983).
7. N. N. Konobeeva and M. B. Belonenko, *Phys. B: Condens. Matter* **456**, 115 (2014).
8. R. Gerritsma, G. Kirchmair, F. Zähringer, E. Solano, R. Blatt and C. F. Roos, *Nature* **463**, 68 (2010).
9. N. N. Konobeeva, A. A. Polunina and M. B. Belonenko, *Int. J. Theor. Phys., Group Theory Nonlinear Opt.* **17**, 197 (2015).
10. I. I. Cotaescu, C. Crucean and C. A. Sporea, *Eur. Phys. J. C* **76**, 102 (2016).
11. N. D. Birrell and P. C. W. Davis, *Quantum Fields in Curved Space* (Cambridge Univ. Press, 1982).
12. S. R. Das, G. W. Gibbons and S. D. Mathur, *Phys. Rev. Lett.* **78**, 417 (1997).
13. L. D. Landau and E. M. Lifshitz, *Quantum Mechanics* (Pergamon Press, 1965).
14. G. Beitme and A. Erdein, *Higher Transcendental Functions* (Nauka, 1973).