1. (a) Let $X$ be the outcome of rolling a fair die. What is the probability generating function (pgf) of $X$? Use the pgf to find $E_X, Var X$.

(b) Toss a die repeatedly. Let $\mu_n$ be the number of ways to throw the die until the sum of the faces is $n$. (So $\mu_1 = 1$ (first throw equals 1), $\mu_2 = 2$ (either first throw is 2, or the first two throws give 1 each), and so on). Find the generating function of $\mu_n, n \geq 1$.

2. A coin shows heads with probability $p$. Let $X_n$ be the number of flips required to obtain a run of $n$ consecutive heads. Show that $E(X_n) = \sum_{i=1}^n p^{-i}$.

3. Let $\{X_n : n \geq 1\}$ be iid uniformly distributed random variables on $[0, 1]$. Let $0 < x < 1$ and define

$$N := \min\{k \geq 1 : X_1 + \ldots X_k > x\}.$$ 

Show that $\Pr(N > n) = x^n/n!$. Find the expectation and variance of $N$.

4. Suppose $X$ follows a Poisson distribution with parameter $\Lambda$ where $\Lambda$ is exponential with parameter $\mu$. Show that $X$ follows a geometric distribution.

5. Consider a (Galton-Watson) branching process with offspring distribution given by a geometric mass function $a_k = qp^k, k \geq 0$ where $p + q = 1$. Let $Z_n$ be the size of the $n^{th}$ generation. Let $T = \inf\{n \geq 1 : Z_n = 0\}$ be the extinction time and $Z_0 = 1$.

(a) Find $\Pr(T = n), n \geq 0$.

(b) For what value of $p$ is $E_T < \infty$?

6. A skip free negative random walk. Suppose $\{X_n : n \geq 1\}$ are iid with

$$\Pr(X_n = j - 1) = p_j, \quad j = 0, 1, 2, \ldots$$

and

$$\sum_{j=0}^\infty p_j = 1, \quad \phi(s) = \sum_{j=0}^\infty p_j s^j, \quad 0 \leq s \leq 1.$$ 

Define $S_0 = X_0 = 1$ and for $n \geq 1$,

$$S_n = X_0 + X_1 + \ldots + X_n.$$
(So the random walk starts at 1; when it moves in the negative direction, it does
so only by jumps of -1.) Let

$$N = \inf\{n \geq 1 : S_n = 0\}$$

denote the first time the random walk hits the level 0. If $P(s) = \mathbb{E} s^N$, show that $P(s) = s\phi(P(s))$.

7. A die is rolled repeatedly. Which of the following are Markov Chains. For those
that are, supply the transition matrix.

(a) The largest number $X_n$ shown upto the $n^{\text{th}}$ roll.
(b) The number $N_n$ of sixes in $n$ rolls.
(c) At time $r$, the time $C_r$, since the most recent six.
(d) At time $r$, the time $B_r$, until the next six.

8. For a Markov Chain $\{X_n : n \geq 0\}$ on a countable state space $S$ prove that

$$\mathbb{P}(X_n = j | X_{n_1} = i_1, \ldots, X_{n_k} = i_k) = \mathbb{P}(X_n = j | X_{n_k} = i_k)$$

whenever $0 \leq n_1 < n_2 < \ldots < n_k < n$ and $i_1, \ldots, i_k, j \in S.$