Verifying Linearizability via Optimized Refinement Checking

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Abstract—Linearizability is an important correctness criterion for implementations of concurrent objects. Automatic checking of linearizability is challenging because it requires checking that 1) all executions of concurrent operations are serializable, and 2) the serialized executions are correct with respect to the sequential semantics. In this work, we describe a method to automatically check linearizability based on refinement relations from abstract specifications to concrete implementations. The method does not require that linearization points in the implementations be given, which is often difficult or impossible. However, the method takes advantage of linearization points if they are given. The method is based on refinement checking of finite-state systems specified as concurrent processes with shared variables. To tackle state space explosion, we develop and apply symmetry reduction, dynamic partial order reduction, and a combination of both for refinement checking. We have built the method into the PAT model checker, and used PAT to automatically check a variety of implementations of concurrent objects, including the first algorithm for scalable non-zero indicators. Our system is able to find all known and injected bugs in these implementations.

Index Terms—Linearizability, Refinement, Model Checking, PAT.

1 INTRODUCTION

Linearizability [31] is an important correctness criterion for implementations of objects shared by concurrent processes, each of which performs a sequence of operations on the shared objects. Informally, a shared object is linearizable if each operation on the object can be understood as occurring instantaneously at some point, called the linearization point, between its invocation and its response, and its behavior at that point is consistent with the specification for the corresponding sequential execution of the operation.

One common strategy (used in manual proofs and automatic verification) for proving linearizability is to determine linearization points in the implementations of all operations and then show that these operations are executed atomically at the linearization points [20], [3], [58]. However, for many concurrent algorithms (e.g., the elimination backoff stack [29], the restricted double-compare single-wrap operation [28], the Herlihy and Wing queue [31], the optimized version [19] of Michael and Scott’s lock-free FIFO queue [42], the fine-grained set with wait-free contains operation [58] and the scalable non-zero indicators [21]), it is difficult or even impossible to statically determine all linearization points. Taking a particular example, in the K-valued register algorithm (Section 10.2.1 of [5]), linearization points differ depending on the execution history. Furthermore, the linearization points determined might be incorrect, which can lead to wrong verification results. Therefore, it is desirable to have automatic methods for verifying these algorithms without the knowledge of linearization points. However, existing methods for automatic verification without using linearization points either apply to limited kinds of concurrent algorithms [60] or are inefficient [58].

In this work, linearizability is defined as trace refinement of operation invocations and responses from a specification to an implementation, where the specification is correct with respect to sequential semantics. Trace refinement (hereafter refinement) is a subset relationship between traces of two systems. That is, a concrete implementation refines an abstract
specification if and only if the set of execution traces of the implementation is a subset of those of the specification. The idea of casting linearizability as refinement has been explored before. Alur et al. showed that linearizability can be cast as containment of two regular languages [2]. Derrick et al. expressed linearizability as non-atomic refinement between Object-Z and CSP models [14]. Other approaches prove linearizability of various algorithms using trace simulation [10], [19], [40].

Our method is not limited to any particular kinds of modeling languages or concurrent algorithms. It exploits model checking of finite state systems specified as concurrent processes with shared variables. In particular, linearizability is verified using refinement checking methods. Though sometimes practically feasible [46], the worst-case execution time of refinement checking is exponential in the size of the abstract specification. To handle real-world concurrent objects, we exploit powerful optimizations to improve the efficiency and scalability of our refinement checking algorithm.

Firstly, our refinement checking explores system behaviors on-the-fly so that a counterexample, if it exists, is produced without generating the entire state space.

Secondly, we combine state-of-the-art state reduction techniques to combat the state space explosion. The first one is symmetry reduction. Symmetry reduction targets at a system composed of sets of behaviorally similar or identical components. Such similarity, or symmetry, often induces equivalent portions of the underlying state space of a system. Provided that a property to be checked whose satisfaction remains unchanged at each equivalent state, exploring one state among the equivalent states is sufficient for verifying the property. A system that models multiple processes manipulating a shared object concurrently tends to exhibit a high degree of symmetry, since each operation on this object often originates from a generic system description without distinguishing the processes. Usually either all processes are symmetric, or they can at least be divided into several classes of symmetric processes. For example, in the algorithm for mutual exclusion without priority, each process competing for the access to critical section is equivalent to one another, and thus this system exhibits full symmetry; in the readers-writers protocol, all readers have the same behavior, and so they are “interchangeable” whereas readers and writers cannot be interchanged. Therefore, the readers-writers protocol contains symmetry in readers, but the global behavior is asymmetric. Based on this observation, we apply the symmetry reduction technique for refinement model checking [43] to exploit symmetry between similar processes to reduce the state space.

The second one is partial order reduction. Concurrently executing processes generate different interleaving traces, and these traces often produce equivalent behaviors. The intuitive idea of partial order reduction is to explore only one interleaving ordering of equivalent traces. In practice, many concurrent object algorithms are designed to minimize the costs of inter-process communication and coordination for scalability reason, by reducing the granularity and frequency of locking. Due to the loose coupling between processes, processes potentially have a number of independent steps, and partial order reduction can be fairly efficient. Because pointer variables are frequently used in these algorithms, static approaches fail to accurately detect the independence between program statements. Thus we apply a dynamic approach called cartesian partial order reduction [27] in this work.

Then, we combine the above two optimization techniques (which has never been explored before in refinement model checking algorithms) to achieve maximum reduction. We prove the soundness and completeness of our combination algorithm. Experimental results show that the combination of partial order and symmetry can yield even better reductions in model checking concurrent object algorithms than either of the two techniques alone.

Our method does not rely on the knowledge of linearization points, but can take advantage of them if given. If linearization points are given (e.g., marked in the implementation), our method constructs an even smaller search space. Some of the optimization techniques are specialized for linearizability checking while others are general. The result is a powerful linearizability checking method that is much more efficient than our prior work [37].

We extend the PAT model checker [51], [38] to support the proposed approach. PAT supports an event-based modeling language [50] that has a rich set of concurrent operators. We apply the proposed method to automatically check finite-state implementations of concurrent object algorithms, such as concurrent counter and queue algorithms, complicated objects with external garbage collector, such as concurrent list-based set [59], as well as sophisticated algorithms—this work is the first published formal verification of scalable non-zero indicators [20] and the mailbox problem [4]. Both algorithms use sophisticated data structures and control structures, and therefore the linearization points are difficult to determine. Counterexamples were reported quickly for incorrect algorithms, such as an incorrect implementation of concurrent queues [47]. Experimental results confirm that our method with the new optimizations is significantly more efficient and scalable than our prior results [37] and other work [58]. Note that our

2. This algorithm is omitted in this paper due to its size. The details can be found in PAT built-in examples.
method only verified the finite-state versions of these algorithms.

The rest of the article is structured as follows. Section 2 presents the definition of linearizability. Section 3 shows how to cast linearizability as refinement relations and proves its correctness. Section 4 describes the verification algorithm and the optimization methods. Section 5 presents experimental results. Section 6 discusses related work. Section 7 concludes.

2 LINEARIZABILITY

Linearizability [31] is a safety property of concurrent systems, over sequences of actions corresponding to the invocations and responses of the operations on shared objects. We begin by formally defining the shared memory model.

Definition 1 (System Models). A shared memory model \( M = (O, \text{init}_O, P) \) is a 3-tuple structure \( (O, \text{init}_O, P) \), where \( O \) is a finite set of shared objects, \( \text{init}_O \) is the initial valuation of \( O \), and \( P \) is a finite set of processes accessing the objects.

Every shared object has a set of states that it could be in. Each shared object supports a set of operations, which are pairs of invocations and matching responses. These operations are the only means of reading or writing the state of the object. A shared object is deterministic if, given the current state of the object and an invocation of an operation, the next state of the object, as well as the return value of the operation, are unique. Otherwise the shared object is non-deterministic. A sequential specification\(^3\) of a deterministic (resp. non-deterministic) shared object is a function that maps every pair of invocation and object state to a pair (resp. a set of pairs) of response and a new object state.

Formally, an execution of a shared memory model \( M = (O, \text{init}_O, P) \) is modeled by a history, which is a sequence of operation invocations and response actions that can be performed on \( O \) by processes in \( P \). The behavior of \( M \) is defined as the set, \( H \), of all possible histories together. A history \( \sigma \in H \) induces an irreflexive partial order \(<_\sigma\) on operations such that \( op_1 <_\sigma op_2 \) if the response of operation \( op_1 \) occurs in \( \sigma \) before the invocation of operation \( op_2 \). Operations in \( \sigma \) that are not related by \(<_\sigma\) are concurrent. A history \( \sigma \) is sequential iff \(<_\sigma\) is a strict total order. Let \( \sigma |_{p_i} \) be the projection of \( \sigma \) on process \( p_i \), which is the subsequence of \( \sigma \) consisting of all invocations and responses that are performed by \( p_i \) in \( P \).

\( \sigma |_{o_i} \) be the projection of \( \sigma \) on object \( o_i \) in \( O \), which is the subsequence of \( \sigma \) consisting of all invocations and responses of operations that are performed on object \( o_i \). Every history \( \sigma \) of a shared memory model \( M = (O, \text{init}_O, P) \) must satisfy the following basic properties:

- **Correct interaction**: For each process \( p_i \) in \( P \), \( \sigma |_{p_i} \) consists of alternating invocations and matching responses, starting with an invocation. This property prevents pipelining\(^4\) operations.

- **Closedness**: Every invocation has a matching response. This property prevents pending operations.

A sequential history \( \sigma \) is legal if it respects the sequential specifications of the objects. More specifically, for each object \( o_i \), there exists a sequence of states \( s_0, s_1, s_2, \ldots \) of \( o_i \), such that \( s_0 \) is the initial valuation of \( o_i \), and for all \( j = 1, 2, \ldots \) according to the sequential specification (the function), the \( j \)-th invocation in \( \sigma |_{o_i} \) together with state \( s_{j-1} \) will generate the \( j \)-th response in \( \sigma |_{o_i} \) and state \( s_j \). For example, a sequence of read and write operations of an object is legal if each read returns the value of the preceding write if there is one, and otherwise it returns the initial value.

Given a history \( \sigma \), a sequential permutation \( \pi \) of \( \sigma \) is a sequential history in which the set of operations as well as the initial states of the objects are the same as in \( \sigma \). The formal definition of linearizability is given as follows.

Definition 2 (Linearizability). Given a model \( M = (O = \{o_1, \ldots, o_k\}, \text{init}_O, P = \{p_1, \ldots, p_n\}) \). Let \( H \) be the behavior of \( M \). \( M \) is linearizable if for any history \( \sigma \) in \( H \), there exists a sequential permutation \( \pi \) of \( \sigma \) such that

1) for each object \( o_i \) (1 ≤ \( i \) ≤ \( k \)), \( \pi |_{o_i} \) is a legal sequential history (i.e., \( \pi \) respects the sequential specification of the objects), and

2) for every \( op_1 \) and \( op_2 \) in \( \sigma \), if \( op_1 <_\sigma op_2 \), then \( op_1 <_\pi op_2 \) (i.e., \( \pi \) respects the run-time ordering of operations).

Linearizability can be equivalently defined as follows. In every history \( \sigma \), if we assign increasing time values to all invocations and responses, then every operation can be shrunk to a single time point between its invocation time and response time such that the operation appears to be completed instantaneously at this time point [40], [5]. This time point is called its **linearization point**.

Linearizability is defined in terms of the interface (invocations and responses) of high-level operations. In a real concurrent program, the high-level operations are implemented by algorithms on concrete

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3. More rigorously, the sequential specification is for a type of shared objects. For simplicity, however, we refer to both actual shared objects and their types interchangeably in this paper.

4. Pipelining operations mean that after invoking an operation, a process invokes another (same or different) operation before the response of the first operation.

5. This property is not required in the original definition of linearizability in [31]. However adding it will not affects the correctness of our result because by Theorem 2 in [31], for a pending invocation in a linearizable history, we can always extend the history to a complete one and preserve linearizability. We include this property to obviate the discussion for pending linearizabilities.
shared data structures, e.g., a linked list that implements a shared stack object [53]. Therefore, the execution of high-level operations may have complicated interleaving of low-level actions. Linearizability of a concrete concurrent algorithm requires that, despite complicated low-level interleaving, the history of high-level invocation and response actions still has a sequential permutation that respects both the runtime ordering among operations and the sequential specification of the objects. This idea is formally presented in the next section using refinement relations.

Linearizability is a safety property [40], so its violation can be detected with a finite prefix of an execution history. However, liveness properties are also important for some critical systems, which guarantees the progress of the systems. Even if the model satisfies linearizability, it might not progress as desired. For instance, even under a fair scheduler, Treiber’s push/pop [53] might never terminate if there is always another concurrent push/pop. This issue is a known problem of lock-free or nonblocking algorithms (e.g., lock-free stacks [53], [12] and lock-free queue [19], [11]). It in fact reflects a deliberate design choice to give up guaranteed termination of individual operations in favor of a weaker guarantee of overall progress in order to obtain an efficient implementation. This suggests that linearizability is just one of many criteria properties for concurrent object design. We remark that liveness properties can be formulated as Linear Temporal Logic (LTL) formulae (an example is given at the end of Example 1) and checked using standard LTL model checkers (with or without the assumption of a fair scheduler [52], [51]). PAT model checker supports LTL model checking with a number of different fairness assumptions [52], [51].

3 Linearizability as Refinement

In this section, we define system behaviors as labeled transition systems (LTS) and linearizability as a refinement relationship between two system models (or equivalently two LTSs). We propose two ways of constructing the refinement relation, for the case that linearization points are not given and the case that they are given, respectively.

3.1 Semantic Model

First of all, we introduce labeled transition systems as the semantic models used to capture the behaviors of shared memory models defined by high-level operations or representing real concurrent programs.

Definition 3 (Labeled Transition System). A Labeled Transition System (LTS) is a tuple \( L = (S, init, Act, \rightarrow) \) where \( S \) is a finite set of states; \( init \in S \) is an initial state; \( Act \) is a finite set of actions; and \( \rightarrow \subseteq S \times Act \times S \) is a labeled transition relation.

3.2 Linearizability without Linearization Points

In this subsection, we make no assumption on the knowledge of the linearization points, which can be known or unknown. We show how to create high-level linearizable specifications and how to define linearizability as refinement. Linearizability is a local

For simplicity, we write \( s \overset{\alpha}{\rightarrow} s' \) to denote \((s, \alpha, s') \in \rightarrow\). The set of enabled actions at \( s \) is \( \text{enabled}(s) = \{ \alpha \in Act \mid \exists s' \in S, s \overset{\alpha}{\rightarrow} s' \} \). A path \( \pi \) of \( L \) is a sequence of alternating states and actions, starting and ending with states \( \pi = (s_0, \alpha_1, s_1, \alpha_2, \cdots) \) such that \( s_0 = \text{init} \) and \( s_i \overset{\alpha_i}{\rightarrow} s_{i+1} \) for all \( i \). If \( \pi \) is finite, then \( |\pi| \) denotes the number of transitions in \( \pi \). A path can also be infinite, i.e., containing infinite number of actions. Since the number of states are finite, infinite paths are paths containing loops. The set of all possible paths for \( L \) is written as \( \text{paths}(L) \).

A transition label can be either a visible action or an invisible one. Given an LTS \( L \), the set of visible actions in \( L \) is denoted by \( \text{vis}_L \) and the set of invisible actions is denoted by \( \text{invis}_L \). A \( \tau \)-transition is a transition labeled with an invisible action. A state \( s' \) is reachable from state \( s \) if there exists a path that starts from \( s \) and ends with \( s' \), denoted by \( s \Rightarrow s' \). The set of \( \tau \)-successors is \( \tau(s) = \{ s' \in S \mid s \Rightarrow s' \wedge \alpha \in \text{invis}_L \} \).

The set of states reachable from \( s \) by performing zero or more \( \tau \) transitions, denoted as \( \tau^*(s) \), can be obtained by repeatedly computing the \( \tau \)-successors starting from \( s \) until a fixed point is reached. We write \( s \Rightarrow s' \) iff \( s' \) is reachable from \( s \) via only \( \tau \)-transitions, i.e., there exists a path \((s_0, \alpha_1, s_1, \alpha_2, \cdots, s_n)\) such that \( s_0 = s, s_n = s' \) and \( s_i \overset{\alpha_i}{\rightarrow} s_{i+1} \land \alpha_i \in \text{invis}_L \) for all \( i \) . Given a path \( \pi \), we can obtain a sequence of visible actions by omitting states and invisible actions. The sequence, denoted as \( \text{trace}(\pi) \), is a trace of \( L \). The set of all traces of \( L \), is written as \( \text{traces}(L) = \{ \text{trace}(\pi) \mid \pi \in \text{paths}(L) \} \).

LTSs can often be shown graphically, e.g., Fig. 1 shows an example LTS\(^6\), where invisible transition labels are omitted for simplicity. We define the refinement relation between two LTSs, usually called trace refinement, as follows.

Definition 4 (Refinement). Let \( L_1 \) and \( L_2 \) be two LTSs. \( L_1 \) refines \( L_2 \), written as \( L_1 \sqsubseteq_T L_2 \) iff \( \text{traces}(L_1) \subseteq \text{traces}(L_2) \).

6. The dotted circles will be explained in Section 4.
property, i.e., a system is linearizable iff each individual shared object is linearizable. Hence we assume there is only one shared object in a system without loss of generality.

3.2.1 Linearizable Specification

To create a high-level linearizable specification for a shared object, we rely on the idea that, in any linearizable history, any operation can be thought of as occurring at some linearization point. For a shared object \( o \), we define a specification model \( M_{sp} = \{ \langle o \rangle, init(o), \mathcal{P}_{sp} \} \) as follows. Every execution of an operation \( op \) of \( o \) on a process \( p_i \in P_{sp} \) includes three atomic steps: the invocation action \( inv(op) \), the linearization action \( lin(op) \), and the response action \( res(op, resp) \). Since there is only one object \( o \), we omit the superscript \( o \) for simplicity. The linearization action \( lin(op)_i \) performs the computation based on the sequential specification of the object. In particular, it maps the invocation and the object state before the operation to a new object state and a response, changes the object to the new state, and stores the response \( resp \) locally. The response action \( res(op, resp)_i \) generates the actual response \( resp \) using the stored result from the linearization action. Each of the three actions is executed atomically without interruption. However, the three actions of an operation may be interleaved with actions of other operations being performed by other processes.

Given a deterministic shared object \( o \), the corresponding model \( M_{sp} = \{ \langle o \rangle, init(o), \mathcal{P}_{sp} \} \) can be constructed as follows. Each operation \( op \) that can be performed by process \( p_i \in P_{sp} \) is defined as a circular state machine with three states (a) an idle state \( s_{p_i,0} \), (b) a state \( s_{op,p_i,1} \) after the invocation of \( op \) but before the linearization action of \( op \), and (c) a state \( s_{op,resp,p_i,2} \) for every response \( resp \) of \( op \), representing the state after the linearization action of \( op \) but before the response of \( op \). The transition from state \( s_{p_i,0} \) to state \( s_{op,p_i,1} \) is triggered by an invocation action \( inv(op) \). The transition from state \( s_{op,p_i,1} \) to state \( s_{op,resp,p_i,2} \) is triggered by an linearization action \( lin(op) \). The transition from state \( s_{op,resp,p_i,2} \) to state \( s_{p_i,0} \) is triggered by an response action \( res(op, resp) \). We let all invocation actions and response actions be visible actions and all linearization actions be invisible actions. This is because only the invocation and response are considered in the behavior of \( M_{sp} \) (refer to Section 2 for the definition of behavior). Each process is defined as the non-deterministic choice of invoking all the allowed operations on object \( o \).

Let \( L_{sp} = \langle S_{sp}, init_{sp}, Act_{sp}, \rightarrow_{sp} \rangle \) be the semantic model of \( M_{sp} \). Then \( S_{sp} \) is the cross product of all object values and all process states. Initial state \( init_{sp} \) is the combination of the initial values of object \( o \) and \( s_{p_i,0}'s \) for all processes \( p_i \). For \( s \in S_{sp} \), let \( s_{vo} \) be the value of object \( o \) encoded in \( s \), \( s_{pi} \) be the state of \( p_i \) in \( s \), and \( s_{p_i,v_o} \) and \( s_{p_i,v_o} \) be the state \( s \) excluding \( s_{vo} \) and excluding \( s_{pi} \) and \( s_{v_o} \), respectively. The labeled transition relation \( \rightarrow_{sp} \) is such that for \( \langle s, e, s' \rangle \in \rightarrow_{sp} \) (a) if \( e = inv(op)_i \), then \( s_{p_i} = s_{p_i,v_o} = s_{p_i,0} \) and \( s_{p_i} = s_{op,p_i,1} \); (b) if \( e = lin(op)_i \), then \( s_{-p_i,v_o} = s_{-p_i,v_o} \), \( s_{p_i} = s_{op,p_i,1} \), and \( s_{p_i} = s_{op,resp,p_i,2} \), where \( s_{vo} \) and \( resp \) are the new object value and the response, respectively, based on the sequential specification of object \( o \) and the old state \( s_{vo} \) and the state \( s_{p_i} = s_{op,p_i,1} \) of process \( p_i \); (c) if \( e = res(op, resp)_i \), then \( s_{p_i} = s_{p_i,v_o} \) and \( s_{p_i} = s_{op,resp,p_i,2} \) and \( s_{p_i} = s_{p_i,0} \).

To create a high-level linearizable specification for a non-deterministic shared object, the same idea above applies. Assume the object \( o \) has \( j \) non-deterministic response values after invoking operation \( op \). Formally, each operation \( op \) that can be performed by process \( p_i \) is defined as a state machine with \( (j + 2) \) states: (a) an idle state \( s_{p_i,0} \), (b) a state \( s_{op,p_i,1} \) after the invocation of \( op \) but before the linearization action of \( op \), and (c) \( j \) states \( \{ s_{op,resp1,p_i,2}, \ldots s_{op,resp_j,p_i,2} \} \) for all possible responses of \( op \), representing the states after the linearization action of \( op \) but before the response of \( op \). Circular state transitions are similarly defined as well.

3.2.2 Implementation Formalization

To formalize any given concurrent algorithm, we need to identify the invocation and response actions so that histories can be formulated based on these actions. For a concurrent algorithm implementing a shared object \( o \), we define an implementation model \( M_{im} = \{ \langle o \rangle, init_{im}, \mathcal{P}_{im} \} \). We assume that \( P_{im} \) is a parallel execution of all processes. Furthermore, the behavior of each process is assumed to be infinite non-deterministic invocations of all the operations supported by the object. In this work, we use \( Process(i) \) and \( System \) to model the behaviors of process \( P_i \) and the algorithm respectively. One example can be found in Algorithm 3.3. Each operation is defined by the algorithm. Executions of program statements shall be considered as (invisible) actions. The atomicity of a statement execution can be defined based on the actual hardware architectures. For example, if a computer architecture can compute \( x = x + 1 \) in one step, then we can treat this statement as atomic action under this particularly architecture. Taking another example, if a computer architecture supports an atomic compare-and-swap (CAS) instruction, then executing the statements representing a CAS is an atomic action under this particular architecture. Further, local statements can be grouped into one atomic

7. Compare-and-swap (CAS) is an atomic CPU instruction used in multithreading to achieve synchronization. It compares the contents of a memory location to a given value and, only if they are the same, modifies the contents of that memory location to a given new value. The atomicity guarantees that the new value is calculated based on up-to-date information; if the value had been updated by another thread in the meantime, the write would fail.
Algorithm 3.1 K-valued register specification

\[
\begin{array}{ll}
\text{shared } R := 0; & \text{Procedure write}(v) \\
\text{Procedure read} & 1: \text{ } v := R; \\
& 1: \text{ } R := v; \\
& 2: \text{ } \text{return } v \\
& 2: \text{ } \text{return }
\end{array}
\]

Reader := read; Reader
Writer := (write(0) \ [8 \ldots 1 \text{ write}(K - 1)); Writer
System := Reader || Writer

Algorithm 3.2 K-valued register implementation

Initially the shared registers \( B[0] \) through \( B[K-1] \) are all 0.

\[
\begin{array}{ll}
\text{Procedure read} & 1: \text{ } i := 0; \\
& 1: \text{ } B[i] := 1; \\
& 2: \text{ } \text{while } B[i] = 0 \text{ do} \\
& 2: \text{ } \text{for } i := v - 1 \text{ downto } 0 \\
& 2: \text{ } \text{do} \\
& 3: \text{ } \text{end while} \\
& 3: \text{ } B[i] := 0; \\
& 5: \text{ } up, v := i; \\
& 5: \text{ } \text{return }
\end{array}
\]

Reader := read; Reader
Writer := (write(0) \ [\ldots] \text{ write}(K - 1)); Writer
System := Reader || Writer

action to reduce the state space of LTS. This can be considered as manual partial order reduction. Since
we are interested in the histories of the algorithm, an invocation action is added to the beginning of each
operation and a response action is added to every return statement of each operation. All other actions
(i.e., the statement execution) inside the algorithm are treated as invisible action, since they do not contribute
to the histories. The semantic model of \( M_{im} \) is denoted by an LTS \( L_{im} = (S_{im}, \text{init}_{im}, \text{Act}_{im}, \rightarrow_{im}) \).

In the following, we use a K-valued register implementation to demonstrate how our linearizability
checking approach works.

Example 1. K-valued register (Section 10.2.1 of [5])

A K-valued \((K > 2)\) single-writer single-reader register \( R \) can be simulated using an array \( B \) of \( K \)
binary single-writer single-reader registers. The possible values of \( R \) are \( \{0, 1, \ldots, K - 1\} \). The value \( i \)
is represented by a 1 in the \( i^{th} \) entry of array \( B \) and 0 in all other entries. For each binary register, there is a
single processor (the writer) that can write to it and a single processor (the reader) that can read from it,
and the values read or written can only be 0 or 1.

When read and write operations do not overlap, it is simple to perform the operations, i.e., a read operation
scans the array beginning with index 0 until it finds a 1 in some entry and returns the index of this entry.
A write operation writes the value \( v \), by writing the value 1 in the entry whose index is \( v \) and clearing
(setting to 0) the entry corresponding to the previous value, if different from \( v \).

When read and write operations might overlap, to ensure that a read operation reads the last value
written, two changes are made: A) a write operation clears only the entries whose indices are smaller than
the value it is writing, and B) a read operation does not return when it finds the first 1 but makes sure that
all lower indexed bits are still zero. Specifically, the reader scans from the low indices toward the high
indices until it finds the first 1; then it reverses direction and scans back down to the beginning, keeping
track of the smallest index observed to contain a 1 during the downward scan. This is the value returned.
Details are given in as Algorithm 3.2. Note that the linearization point of the read operation is not fixed
because the value of \( v \) to be returned depends on the last position of value 1 found during the downwards
scan. i.e., only the last execution of line 8 is the linearization point rather than any execution of line
8 is the linearization point.

The linearizable specification model is defined as in Algorithm 3.1, where \( R \) is the shared register with
initial value 0. The statement in Line 1 for both read and write operations is the linearization action.

The refinement relation is based on comparing the arguments of the invocation and return values of the
responses of an operation. In this example, the only visible actions are the invocation and response of
the read and write operations. The system model is constructed as a general client on the shared register,
which consists of a parallel composition of a reader process and a writer process. Further, the writer process
non-deterministically chooses one of \( K \) possible values to execute write operation. If there are multiple
operations a process can perform, then a process is modeled to non-deterministically execute one of
the operations.

We remark that interesting progress properties can be verified by model checking. For example,
suppose \( K = 3 \), then one can model check the formula \( \square \text{ inv(read)} \rightarrow (\diamond \text{ res(read,0)} \lor \text{ res(read,1)} \lor \text{ res(read,2)}) \), where \( \square \) and \( \diamond \)
are modal operators denoting ‘always’ and ‘eventually’ respectively. This property says that once reader
invokes the read operation, it will eventually get the value rather than blocked by the writer indefinitely.

3.2.3 Linearizability Formalization

The following theorem characterizes linearizability of an implementation \( M_{im} \) through a refinement
relation, where the corresponding specification \( M_{sp} \) is created by following the steps in Section 3.2.1, and
8. \[ \square \] denotes nondeterministic choice between operations.
9. \[ || \] denotes the parallel composition of processes.
The above theorem shows that in order to verify linearizability of an implementation, it is necessary and sufficient to show that the implementation LTS is a refinement of the specification LTS. This provides the theoretical foundation of our method. Notice that the verification by refinement given above does not require identifying low-level actions in the implementation as linearization points, which can be difficult (or even impossible) for some algorithms (e.g., the elimination backoff stack [29], the restricted double-compare single-wrap operation [28]). In fact, the verification can be automatically carried out without special knowledge about the implementation beyond the implementation code itself.

3.3 Linearizability with Linearization Points

In some cases, one may be able to identify certain actions in an implementation as linearization points, which are linearization actions. This subsection presents an alternative and simpler way of formalization when the linearization points are known.

3.3.1 Linearizable Specification

When the linearization points are known, a linearizable specification can be constructed in the similar way as in Section 3.2.1. The difference is that we make linearization actions visible and hide the invocation and response actions. More specifically, we obtain a specification LTS \( L_{sp} \) by the following two modifications to \( L_{sp} \): (a) we change each linearization action \( lin(op) \) to \( lin(op, resp) \) to include the response \( resp \) computed by this linearization action; This is possible because the return value of an operation is available after linearization action. and (b) we make only linearization actions visible and all \( inv(op) \) and \( res(op, resp) \) invisible.

3.3.2 Implementation Formalization

To formalize any given concurrent algorithms in this case, we adopts the same way as for specification construction. After generating the \( L_{im} \) by following 3.2.2, we mark only linearization actions visible and hide all other actions as above. There is no need to add invocation and response actions in this case. Similar to the construction of linearizable specifications, we include the responses in the linearization actions. We demonstrate this idea using the following example.

Example 2. Abstract Concurrent Counter

Treiber [53] proposed a concurrent stack implementation using compare-and-swap (CAS) instructions. Here we use one of its simplified versions presented in [10], as shown in Algorithm 3.3. The pointer \( H \) is shared by all processes. Each operation tries to update \( H \) until its CAS operation succeeds. To keep this algorithm finite-state (hence subject to model
Algorithm 3.3 Concurrent stack implementation

type Node = {val : T; next : Node};
shared Node H := null;
N is the maximum value that can be stored by the stack

Procedure push
1: n := new Node();
2: n.val := v;
3: return
4: ss := H;
5: n.next := ss;
6: until CAS(H, ss, n)
7: return
8: Procedure pop
1: repeat
2: ss := H;
3: if ss = null then
4: return empty
5: end if
6: n := ss.next;
7: lv := ss.val;
8: until CAS(H, ss, n)
9: return lv

Process(i) := (push(1) \ldots pop(N) \parallel pop); Process(i)
System := Process(1) \parallel Process(2) \parallel \ldots \parallel Process(N)

Algorithm 3.4 Concurrent counter implementation

shared S := 0;

Procedure push
1: repeat
2: ss := H;
3: n := ss + 1;
4: until CAS(H, ss, n)
5: return
6: n := ss - 1;
7: until CAS(H, ss, n)
8: return ss

Process(i) := (push \parallel pop); Process(i)
System := Process(1) \parallel Process(2) \parallel \ldots \parallel Process(N)

Since the concurrent counter implements the standard push and pop operations, the specification of the concurrent counter algorithm is defined in Algorithm 3.4. Here the actual data are abstracted because only the stack size is relevant, which is modeled as shared variable \( S \) with initial value 0. A push operation increases the stack size by 1. A pop operation decreases the stack size by 1 if it is bigger than 0 and returns the stack size before the decrease; otherwise it returns 0. We introduce atomic construct to indicates that the program in the block is to be executed as one super-step, non-interleaved with other processes. This atomic construct is the linearization action for the pop operation.

The linearization points of Algorithm 3.3 are known [3]. Therefore the verification can be conducted directly by modeling linearization points and leaving out invocation and response actions. Clearly the specification model of the stack has only one linearization action (i.e., the corresponding atomic block) for each operation. We make each linearization action visible and include the return value. For push operation, its linearization action is \( \text{lin}(\text{push}, S + 1) \), where \( i \) is the process identifier and \( S + 1 \) is the stack size after update. Likewise, the linearization action of pop operation is \( \text{lin}(\text{pop}, S) \), where \( i \) is also the process identifier, and \( S \) is the stack size before update.

The linearization points of the counter implementation are conditional. For push operation, only a successful CAS (at line 4) is considered to be a linearization point. For pop operation, there are two conditional linearization points: if the counter is 0, returning 0 at line 4 is a linearization point; otherwise, a successful CAS (line 7) is a linearization point.

3.3.3 Linearizability Formalization

Theorem 2. Let \( L'_{im} \) be an implementation LTS with known linearization actions and specified by the steps in Section 3.3.2, and \( L'_{sp} \) be the corresponding specification LTS generated by the steps in Section 3.3.1. All traces of \( L'_{im} \) are linearizable if and only if \( L'_{im} \sqsubseteq_T L'_{sp} \).

Proof:

Sufficient condition: For any trace \( \sigma \in \text{traces}(L'_{im}) \), because \( L'_{im} \sqsubseteq_T L'_{sp} \), \( \sigma \) is also a trace of \( L'_{sp} \). Let \( \rho \) be the execution history of \( L'_{im} \) that generates the trace \( \sigma \). \( \sigma \) respects the sequential specification of the objects since the linearization action in \( L'_{sp} \) is the only action that affects the object states. Furthermore, each linearization action in \( \sigma \) is always between its invocation and response action in \( \rho \) because of the way \( \sigma \) is generated. According to the second definition of linearizability using linearization points in Section 2, \( \sigma \) is the shrank execution of \( \rho \), and each action is the linearization point of the corresponding operation.

Necessary condition: Let \( \sigma \) be a trace of \( L'_{im} \). Let \( \rho \) be the execution history of \( L'_{im} \) that generates the trace \( \sigma \). By assumption, \( L'_{im} \) is linearizable and all linearization actions are identified in the implementation. Therefore \( \sigma \) is the shrank execution of \( \rho \) and each action is the linearization point of the
corresponding operation. Condition “all linearization actions are identified in the implementation” ensures that each operation has a linearization point in $\sigma$. Since linearization actions in $L_{im}$ represent the effects of the object changes, we can find the same linearization action in the $L_{sp}$. Hence $\sigma$ is also a trace of $L_{sp}$.

It is not difficult to see that the implementation model built with the knowledge of linearization points is much simpler and contains fewer visible actions. Hence the verification can be done more efficiently by comparing only one action for each operation. However, it is important to note that, as stated in Theorem 2, to make refinement a necessary condition of linearizability in this case, one has to show that no other actions in the implementation can be linearization points. In other words, the determined linearization points have to be complete. Otherwise, even if the verification finds a counterexample for the refinement relation, it may be due to unidentified linearization points, and one cannot conclude that the implementation is not linearizable.

4 Verification of Linearizability

With the results presented in Section 3, all we need is a scalable refinement checking algorithm in order to establish linearizability. In this section, we present a classic algorithm [45] for refinement checking, and then optimize it with symmetry reduction, partial order reduction, and their combination.

In the following, we fix two LTSs $L_{im} = (S_{im}, \text{init}_{im}, \text{Act}_{im}, T_{im})$ and $L_{sp} = (S_{sp}, \text{init}_{sp}, \text{Act}_{sp}, T_{sp})$, which represent an implementation and a specification respectively. Notice that both $L_{im}$ and $L_{sp}$ typically have invisible actions. That is, if linearization points are unknown, all actions except invocation and response actions are invisible; if linearization points are known, all except the linearization actions are invisible. As a result, both $L_{im}$ and $L_{sp}$ have a degree of non-determinism, e.g., two different processes both can take $\tau$-transitions, resulting in two identically-labeled actions from the same state.

For ease of understanding, we use the bounded abstract concurrent counter algorithm as a running example, and show how the state space of the whole program can be reduced by symmetry reduction, then partial order reduction and at last their combination. Moveover to show generality of our approach, we assume that the linearization points are unknown. To properly display the entire state space, due to space limitations, we only show a stack being used by two processes, with ids 0 and 1, respectively. So the effect of reduction approaches can be demonstrated visually, in order to help readers better understand the technical details of our approach. Furthermore, we require that each process perform only one push operation. The stack is initially empty and its size is 2.

### 4.1 A Linearizability Checking Algorithm

In order to establish a refinement relationship between $L_{im}$ and $L_{sp}$, we need to show that every trace of $L_{im}$ is allowed by $L_{sp}$. Because of non-determinism in $L_{sp}$ after a sequence of visible actions there may be many states that the system might be in. A refinement checking algorithm thus will have to keep track of all the states reachable in $L_{sp}$ on a given trace, which can be achieved by determinization, also known as normalization. A determinization of an LTS $L$ is a deterministic LTS, written as $D(L)$, such that $L$ and $D(L)$ have the same traces. With determinization, checking whether $L_{im}$ refines $L_{sp}$ is reduced to check whether $L_{im}$ refines $D(L_{sp})$, which is easier because there is exactly one state in $D(L_{sp})$ corresponding to each possible trace. A standard approach for determinization is through subset construction [32].

**Definition 5 (Determinization).** Let $L = (S, \text{init}, \text{Act}, \rightarrow)$ be an LTS. The determinized LTS of $L$ is $D(L) = (S_d, \text{init}_d, \text{Act}_d, \rightarrow_d)$ where $S_d \subseteq 2^S$ is a set of subsets of $S$; $\text{init}_d = \tau^*(\text{init})$; $\text{Act}_d = \text{vis}_{D(L)} = \text{vis}_L$ and $\rightarrow_d \subseteq S_d \times \text{Act}_d \times S_d$ is a transition relation such that $X \rightarrow_d Y$ iff $Y = \{y : \exists x \in X, \exists s \in S : x \xrightarrow{a} s \land y \in \tau^*(s)\}$.

In the following we fix $D_{sp} = (S_d, \text{init}_d, \text{Act}_d, T_d)$ to be the determinized LTS of $L_{sp}$. All states connected by $\tau$-transitions in $L_{sp}$, are grouped in $D_{sp}$. 

Algorithm 4.1 A linearizability checking algorithm

**Procedure Linearizability ($L_{im}, L_{sp}$)**

1: $\text{checked} := \emptyset$
2: $\text{pending}.\text{push}((\text{init}_{im}, \tau^*(\text{init}_{sp})))$
3: $\text{stack}.\text{depth}.\text{push}(0)$
4: $\text{path}.\text{depth} := \emptyset$
5: $\text{counterexample} := \emptyset$
6: while $\text{pending} \neq \emptyset$
7: $\text{(im}, \text{sp}) := \text{pending}.\text{pop}()$
8: $\text{checked} := \text{checked} \cup \{(\text{im}, \text{sp})\}$
9: $d := \text{stack}.\text{depth}.\text{pop}()$
10: while $d > 0 \land \text{path}.\text{depth}.\text{peek}() \geq d$
11: $\text{path}\text{.depth}.\text{pop}()$
12: $\text{counterexample}.\text{push}()$
13: end while
14: $\text{path}.\text{depth}.\text{push}(d)$
15: $\text{counterexample}.\text{push}((\text{im})$
16: if $\text{sp} = \emptyset$ then
17: return $\text{counterexample}$
18: end if
19: for all $(\text{im}', \text{sp}') \in \text{next}(\text{im}, \text{sp})$ do
20: if $(\text{im}', \text{sp}') \notin \text{checked}$ then
21: $\text{pending}.\text{push}((\text{im}', \text{sp}'))$
22: $\text{stack}.\text{depth}.\text{push}(d + 1)$
23: end if
24: end for
25: end while
26: return $\text{true}$
For instance, the dotted circles in Fig. 1 show the determined states. It is straightforward to show that \( D_{sp} \) is deterministic, i.e., for any state \( s \in S_d \) and any visible action \( \alpha \in \text{Act}_{d} \), there is at most one state \( s' \in S_d \) such that \( s \xrightarrow{\alpha} d \ s' \).

Refinement checking is then reduced to reachability analysis\(^{10} \) of the synchronous product of \( L_{im} \) and \( D_{sp} \). Each state of the product space is a state pair \((im, sp)\), where \( im \) is an implementation state and \( sp \) is a determined specification state. If there exists a state pair \((im', sp')\) such that \( im \xrightarrow{\alpha} im' \) \( sp \xrightarrow{\alpha} d \ sp' \) for some \( \alpha \in \text{Act}_{d} \cap \text{Act}_{im} \), we say there is a product transition from \((im, sp)\) to \((im', sp')\) labeled with \( \alpha \).

Algorithm 4.1 shows an on-the-fly linearizability checking algorithm. It performs a depth-first-search for a state that violates linearizability, i.e., a pair \((im, sp)\) \( \in S_{im} \times S_{d} \) such that \( sp \) is an empty set. The algorithm returns true if no such pair is found. Otherwise, a counterexample violating trace refinement is found. The algorithm maintains two data structures. \( \text{checked} \) is a set of product states that have been explored and \( \text{pending} \) is a stack containing new states yet to be explored. On line 2, the initial state of the product is pushed into \( \text{pending} \). While there are new states to be explored (i.e., \( \text{pending} \) is not empty), a pending state is obtained from \( \text{pending} \) on line 7, which is then added to \( \text{checked} \) on line 8. If the state is of the form \((im, \emptyset)\), we infer that there exists a trace that leads \( L_{im} \) to state \( s \) and leads \( L_{sp} \) to no state, and therefore the trace serves as a counterexample to refinement. The code fragments for producing a counterexample are shown in grey color, which stores the counterexample only part of \( D_{sp} \) is constructed. In practice, the code fragments are omitted in the algorithms presented later in this section. If a successor state of \((im, sp)\) has not been explored (i.e., \((im', sp') \notin \text{checked} \) on line 20), it is pushed into \( \text{pending} \) on line 21. Note that Function next\((im, sp)\) returns the set of successor states in the product. Formally,

\[
\text{next}(im, sp) = \\
\{ (im', sp') | (im \xrightarrow{\alpha} im' \land \alpha \in \text{invis}_{L_{im}}) \} \cup \\
\{ (im', sp') | \exists \alpha \in \text{vis}_{L_{im}} : im \xrightarrow{\alpha} im' \land \\
\forall x' \in sp', \exists x \in sp, \exists y \in sp' : x \xrightarrow{\alpha} y \land x' \in \tau^*(u) \}
\]

If the implementation takes a \( \tau \)-transition (e.g., any action that is not an invocation or response action if the linearization points are unknown), the trace of the implementation remains the same (as \( \tau \) is always pruned from the trace) and therefore the set of corresponding states in \( L_{sp} \) (which are reached via the same trace) remains the same. In other words, the determined state remains the same. If the implementation takes a visible transition, then the same action must be performed by the specification, resulting in a new determined state. To obtain next\((s, X)\) algorithmically, it is necessary to compute the set of states reached by a \( \tau \)-transition from a given state, which can be implemented using a standard depth-first-search.

For the running example, Fig. 2 shows the determination of its specification LTS and Fig. 3 shows the specification-implementation product LTS explored during linearizability checking (let us ignore for now the colors and shades of edges and nodes). Each node in Fig. 3 contains two parts, the implementation state in the upper part and determined specification state in the lower part. If an action is a visible action, then the corresponding edge is labeled with its name. Otherwise, the edge is labeled with the statement in the push operation of Algorithm 3.4. \( \ell_{i} \) denotes the statement on line \( i \) executed by process \( j \). Because we assume the linearization points of this algorithm are unknown, the visible actions are the invocation and response of the push operation. That is, push_inv.i and push_res.i,v are the invocation and response action respectively where \( i \) is the identifier of the invoking process and \( v \) is the return value of the push operation.

The algorithm terminates as long as the product has finitely many states. The soundness of the algorithm follows from [45]. Note that determination is performed on-the-fly so that in the presence of a counterexample only part of \( D_{sp} \) is constructed. In practice, this algorithm may suffer from state space explosion. Its complexity is linear in the number of transitions in the product. The size of the product is bounded by the size of \( L_{im} \) and \( D_{sp} \). In the worst case, \( D_{sp} \) may have exponentially more states than \( L_{sp} \). Further, \( L_{im} \) and \( L_{sp} \) typically have an exponential blow up in the size of the system models representing the specification and implementation. Thus, it is necessary to explore powerful state reduction techniques in order to model check complex concurrent object implementations.

### 4.2 Optimization 1: Symmetry Reduction

A concurrent data object is often designed to be accessed by many behaviorally similar or even identical processes. Such similarity or symmetry often induces equivalent portions of the underlying state space. Symmetry reduction [24] is an effective technique for eliminating such equivalent states. The idea is to only explore the behavior of one process and conclude the same for homogeneous others (subject to property-specific conditions).

For temporal logic model checking, classical symmetry reduction approaches [34], [48], [23] often choose a unique representative state from each class of equivalent states. Each visited state is replaced with its representative state. Unfortunately, these approaches fail to fit into the context of refinement-style linearizability checking. The significant obstacle is lock-step synchronization on all visible actions between...
the state spaces of implementation and specification models. If symmetry reduction is applied to an implementation (resp. a specification) model, replacing an implementation (resp. specification) state with its representative potentially influences the synchronization result and thus sacrifices the validity of linearizability checking. For instance, suppose a common action result and thus sacrifices the validity of linearizability checking. For instance, suppose a common action

Moffat et al. proposed a symmetry reduction approach for trace refinement checking [43]. In the following, we firstly use their approach to perform symmetry reduction on \( L_{im} \) during linearizability checking. Then we extend and improve their work to achieve better performance.

We begin with introducing relevant notations and terminology. A permutation \( \gamma \) on a finite set of objects (e.g., process identifiers) is a bijection from the set to itself (i.e., a function that is one-to-one and onto). For instance, suppose the system is a parallel composition of \( N \) processes \( P_i \), where \( i \) ranging from 1 to \( N \) is a unique identifier associated with the process. Permutation \( \gamma \) may be defined such that \( \gamma(i) = (i+1) \mod N \) for each \( i \). Suppose \( N = 2 \). We have \( \gamma(1) = 2 \) and \( \gamma(2) = 1 \), written as \((1, 2)\)\(^{11}\). A permutation group is a group of permutations under functional composition \( \circ \). For instance, a permutation group in the above example formed by \( \gamma \) is \( \langle (1, 2), (1, 2) \circ (1, 2) \rangle \), which equals \( \langle (1, 2), (1)(2) \rangle \).

Let \( Perm(S) \) be the group of permutations of the finite set \( S \). Next, we define the concept of automorphism group.

**Definition 6 (Automorphism).** Given an LTS \( L = (S, init, Act, \rightarrow) \), a group \( G \subseteq Perm(s) \times Perm(Act) \) is an automorphism group of \( L \) if and only if

1) \( \forall \gamma \in G, \ s_1, s_2 \in S, \ \alpha \in Act : \ s_1 \xrightarrow{\alpha} s_2 \implies \gamma(s_1) \xrightarrow{\gamma(\alpha)} \gamma(s_2) \)

2) \( \gamma(init) = init \)

It is often the case that an automorphism group is given as a group acting on the process identifiers of the state variables as well as labeled actions. For example, a permutation \( \gamma \), defined on the actions and states in Figure 2, may also be described as the permutation of process identifiers, i.e., \((0, 1)\).

Our starting point of symmetry reduction is the simple observation that, in practice, each operation on a concurrent data object often originates from a generic system description without discriminating process identifiers in both its implementation and specification. This means that if there is any symmetry relation between processes in an implementation model, then there will be the same relation in its

11. Cycle notation is used to write down a permutation such that the contents in parentheses move in a cycle. E.g., \((1, 2)\) denotes moving 1 to 2, and 2 back to 1.
Lemma 1. If a permutation is an automorphism of an LTS $L$, it is an automorphism of $D(L)$.

Proof:
Suppose that permutation $\gamma$ is an automorphism of $L$. By definition, a transition $t = X \overset{\alpha}{\rightarrow} Y$ is in $D(L)$ iff $\exists x \in X, \exists y \in Y : x \overset{\alpha}{\rightarrow} x' \land y \in \tau^*(x')$. Further, $\forall s_1, s_2 \in S, \alpha \in Act : s_1 \overset{\alpha}{\rightarrow} s_2$ iff $\gamma(s_1) \overset{\gamma(\alpha)}{\rightarrow} \gamma(s_2)$. Thus, $x \overset{\alpha}{\rightarrow} x' \Leftrightarrow \gamma(x) \overset{\gamma(\alpha)}{\rightarrow} \gamma(x')$, and $y \in \tau^*(x') \Leftrightarrow \gamma(y) \in \tau^*(\gamma(x'))$. Thus, there is exactly a transition $\gamma(P) \overset{\gamma(\alpha)}{\rightarrow} \gamma(Q)$ in $D(L)$. Considering that $t$ is arbitrary, $\gamma$ is also an automorphism of $D(L)$. \qed

corresponding specification model and vice versa. Note that both the implementation and specification models in this work take the form of a parallel composition of processes. The insight is therefore captured by a premise: a permutation is an automorphism of $L_{im}$ if and only if it is also an automorphism of $L_{sp}$.

Let $G$ be an automorphism group on both $L_{im}$ and $L_{sp}$. Considering that the usual linearizability checking algorithm explores the product space of $L_{im}$ and $D_{sp}$, we need to prove that $G$ is also an automorphism group on this product space so that we can apply symmetry reduction on it. We first prove the following lemma.

**Fig. 3. Specification-implementation product LTS of concurrent stack implementation**
Next, we prove the following theorem:

**Theorem 3.** If $G$ is an automorphism group on both $L_{im}$ and $L_{sp}$, then it is an automorphism group on the product space of $L_{im}$ and $D_{sp}$.

**Proof:**

By Definition 6, we must show that for any $\gamma \in G$,

(i) if $sp \xrightarrow{\alpha_d} sp'$ and $im \xrightarrow{\alpha_{im}} im'$, then $\gamma(sp) \xrightarrow{\gamma(\alpha_d)} \gamma(sp')$ and $\gamma(im) \xrightarrow{\gamma(\alpha_{im})} \gamma(im')$,

(ii) $\gamma(init_d) = init_d$ and $\gamma(init_{sp}) = init_{sp}$.

By the premise, $\gamma$ is an automorphism of $L_{im}$. If $im \xrightarrow{\alpha_{im}} im'$, then $\gamma(im) \xrightarrow{\gamma(\alpha_{im})} \gamma(im')$. By Lemma 1, $\gamma$ is an automorphism of $D_{sp}$. So if $sp \xrightarrow{\alpha_d} sp'$, then $\gamma(sp) \xrightarrow{\gamma(\alpha_d)} \gamma(sp')$ and $\gamma(im) \xrightarrow{\gamma(\alpha_{im})} \gamma(im')$.

Similarly, because $\gamma$ is an automorphism of both $L_{im}$ and $D_{sp}$, $\gamma(init_d) = init_d$ and $\gamma(init_{sp}) = init_{sp}$.

Since $\gamma$ is an arbitrary automorphism in $G$, we conclude that $G$ is an automorphism group on the product space of $L_{im}$ and $D_{sp}$. $\square$

The function $repPair$ is defined to “twist” a state pair in the product space using an automorphism in $G$ as follows:

$$repPair(im, sp, \gamma) = (\gamma(im), \gamma(sp))$$ where $im \in L_{im}$, $sp \in D_{sp}$ and $\gamma \in G$.

A $repPair$-twisted (hereafter twisted) path through the product space is a sequence $\langle s_0, \alpha_1, s_1, \ldots, s_{n-1}, \alpha_n, s_n \rangle$ of states, actions and permutations, starting and ending with states such that: for all $0 \leq i < n$, suppose $s_i = (im, sp)$, there exists a state pair $(im', sp')$ and $\gamma_{i+1} \in G$ such that $im \xrightarrow{\alpha_{im}} im'$, $sp \xrightarrow{\alpha_d} sp'$ and $s_{i+1} = (\gamma_{i+1}(im'), \gamma_{i+1}(sp')) = repPair(im', sp', \gamma_{i+1})$.

In this way, function $repPair$ “twists” the original path explored in the usual refinement checking. For example, there exists a twisted path $\langle (t_1, s_1), push_inv:0, (0, 1), (t_3, s_3), (0, 1), (t_2, s_2), push_inv:1, (0, 1), (t_4, s_6) \rangle$ in Fig. 3.

To generate a representative state for each visiting implementation state, we assume the existence of a function $rep$ such that, for each $s \in S$, $rep(s)$ is a representative of $s$. In order to perform symmetry reduction on $L_{im}$ during linearizability checking, we restrict the definition of $repPair$ using $rep$ as follows:

$$repPair(im, sp, \gamma) = (rep(im), \gamma(sp))$$ such that $\gamma(im) = rep(im)$.

Algorithm 4.2 is developed to perform on-the-fly linearizability checking with symmetry reduction. The underlined text shows the differences compared with Algorithm 4.1. Function $Rep1$ calculates a unique representative of each visited state, i.e., $Rep1(s) = (s', \gamma)$ such that $\gamma(s) = s'$. Each $\gamma$ used in $repPair$ can be stored through the explored path in order to recover a twisted counterexample path to an actual path in $L_{im}$. Assume a linearizability violated pair $s = (im, \emptyset)$ is found, and $\pi$ is the current explored twisted path arriving at $s$. $\pi$ can be easily composed by concatenating all elements popped off the stack $pending$ in reverse order. Let $\pi$ be $\langle s_0, \alpha_1, s_1, \ldots, \alpha_n, s_n \rangle$, then function $recover$ is used to “recover” it to an actual counterexample path in the original product space. Formally the function $recover$ is defined as follows:

$$recover(\pi) = \langle s_0, \alpha_1, \gamma_{-1}(s_1), \ldots, \gamma_{n-1}(s_n) \rangle$$.

The following theorem (which is based on Lemma 2) guarantees the soundness and completeness of Algorithm 4.2. Algorithm 4.2 finds a twisted counterexample path exactly when the refinement does not hold.

**Lemma 2.** [43] Suppose function $repPair$ maps each state pair $(u, v)$ to $(\gamma(u), \gamma(v))$ for some $\gamma \in G$. Then, for all paths $\pi$, there is a path to state pair $s = (im, sp)$ if and only if there is a $repPair$-twisted path $\pi'$ to state pair $\gamma(s) = (\gamma(im), \gamma(sp))$, with $recover(\pi') = \gamma(\pi)$, for some $\gamma \in G$. $\square$

**Theorem 4.** [43] Suppose function $repPair$ maps each state pair $(u, v)$ to $(\gamma(u), \gamma(v))$ for some $\gamma \in G$. $L_{im} \triangleq T D_{sp}$ has a counterexample path $\pi$ if and only if it has a counterexample $repPair$-twisted path $\pi'$ with $recover(\pi') = \pi$. $\square$

Besides the above symmetry reduction, we observe that in practice for a state pair $(im, sp)$, there may exist multiple permutations $\gamma_1, \gamma_2, \ldots, \gamma_n \in G$ such that for every $i$ in $\{1, \ldots, n\}$: $\gamma_i(s) = rep(im)$, and there exists $i, j : i \neq j, \gamma_i(s) \neq \gamma_j(s)$. That is, different permutations may produce different twisted pairs with the same representative implementation state for a state pair in Algorithm 4.2. To check whether counterexample searching is sensitive to the permutations chosen, we develop the following theorem.

**Theorem 5.** For any state $s = (im, sp)$ in the product space of $L_{im}$ and $D_{sp}$, if there exist $\gamma_1, \gamma_2 \in G$ and $\gamma_1 \neq \gamma_2$, then there is a twisted path from $\gamma_1(s)$ to $q$ if and only if there exists a twisted path from $\gamma_2(s)$ to $\gamma(q)$, for some $\gamma \in G$.

**Proof:**

We will prove that there is a twisted path $\pi = \langle s_0 = \gamma_1(s), \alpha_1, \gamma_1(s), \ldots, \alpha_n, s_n \rangle$ if and only if there exists a twisted path $\pi' = \langle s'_0 = \gamma_2(s), \alpha'_1, s'_1, \ldots, \alpha'_n, s'_n \rangle$ such that $q = s_n'$ and for every $0 \leq i \leq n$, there exists $\gamma \in G : \gamma(s_i) = s'_i$.

**Necessary condition: Induction on $|\pi|$.**

**Basis:** $|\pi| = 0$. Then $\pi = \langle \gamma_1(s) \rangle$. There is exactly one twisted path $\pi' = \langle \gamma_2(s) \rangle$. Thus there exists $\gamma = \gamma_1 \gamma_2 \in G : \gamma_1 \gamma(s) = \gamma_2(s)$.

**Induction hypothesis:** Assume that the claim is true for any twisted path $\pi$ such that $|\pi| \leq k$. $\square$
Induction step: We show it also holds for all twisted paths $\pi$ where $|\pi| = k + 1$.

Consider a twisted path $\pi$ of the form $(s_0 = \gamma_1(s), \alpha_1, \gamma_1, \cdots, \alpha_{k+1}, \gamma_{k+1}, s_{k+1})$. Then there is a transition from $s_k$ to $s_{k+1}$ labeled $\alpha$ where $s_{k+1} = \gamma_{k+1}(pre-s_{k+1})$. From the induction hypothesis, there is a twisted path $\pi' = (s'_0 = \gamma_1(s), \alpha_1, \gamma_1, \cdots, \alpha_{k}, \gamma_k, s_{k})$ such that $\gamma(s_k) = s'_k$. By Theorem 3, there is a transition from $s'_k$ to $\gamma_{k+1}(pre-s_{k+1})$ labeled with $\gamma(\alpha)$. So $s'_{k+1} = \gamma_{k+1}(pre-s_{k+1}) = \gamma_{k+1}\gamma_{k+1}(s_{k+1})$.

Sufficient condition: Similar.

An immediate corollary of Theorem 5 is shown as follows, which is the foundation of our improvement.

**Corollary 1.** For any state $s = (im, sp)$ in the product space of $L_{imm}$ and $D_{sp}$, if there exist $\gamma_1, \gamma_2 \in G$ such that $\gamma_1(s) = \gamma_2(s)$ and $\gamma_1 \neq \gamma_2$, then there exists a counterexample twisted path if and only if there exists a twisted path.

Based on the above results, we develop a new algorithm 4.3 to allow more state reduction. The underlined text shows the differences compared with Algorithm 4.2. Function $Rep2$ is defined as: $Rep2(s) = \langle s', \gamma_1, \cdots, \gamma_n \rangle$ such that for every $i \in \{1, \cdots, n\}$: $\gamma_i(s) = s'$. By Corollary 1, it is sufficient to explore only one of the states $(s', \gamma_i(sp))$. Thus if none of these states have been explored on line 11, one state is pushed into pending on line 12.

In the following, we explain how Algorithm 4.3 works step by step on the concurrent stack example.

It is not difficult to find that each pair of states in the symmetric positions are equivalent states in Fig. 2 and Fig. 3, i.e., one can be transformed to the other by process identifier permutation $(1, 2)$, e.g., $(1, 2)(t_3) = t_2$. Further, we assume that each colored state in Fig. 3 is the representative state among the class of its equivalent states.

Starting from the initial state $(s_1, t_1)$, we first nondeterministically pick a transition to execute, the right one for example. At the resultant state $(s_3, t_3)$, we use function $repPair$ to twist it to get a new state to proceed. Because $rep(s_3) = s_2 = (1, 2)(s_3)$, $repPair(s_3, t_3) = \langle rep(s_3), (1, 2)(t_3) \rangle = (s_2, t_2)$. Starting from the product state $(s_2, t_2)$, we again nondeterministically pick one of these transitions, say the right one and get to $(s_4, t_4)$. Because $rep(s_4) = s_4$, there is no need to change $t_4$. Again, if we pick the left transition from $(s_4, t_4)$ and arrive at $(s_2, t_2)$, we twist it to state $(s_6, t_4)$. By proceeding in this manner, we eventually explore all colored states in the product space so that we succeed to perform symmetry reduction on the implementation LTS and preserve the validity of linearizability checking.

### 4.3 Optimization 2: Partial Order Reduction

A process performs an atomic action at each step to move the system from one state to another. Partial order reduction is an effective state space reduction technique for concurrent systems with independent actions. The motivation is that the effect of independent concurrent actions is irrelevant to their interleaving orderings. If the property of interest does not depend on the intermediate states through the execution traces of these actions, a number of equivalent orderings of concurrent actions can be eliminated, which often yields a good reduction on the state space.

In practice, most concurrent object algorithms have a low degree of interprocess interaction and coordination for scalability reason. Many program statements are local computations or accessing disjoint locations of the shared data object. This loose coupling potentially induces many independent transitions and thus
enables effective partial order reduction on these algorithms. In this subsection, we show how to perform partial order reduction for linearizability checking.

We adopt the recently-proposed dynamic partial order reduction technique called cartesian partial order reduction in [27] for linearizability checking, for two reasons. First, pointer variables are used frequently in concurrent object algorithms. Static partial order reductions [35], [57], [26] fail to identify their independence precisely and thus cause a poor reduction on the state space. Second, concurrent algorithms with optimistic or lazy synchronization (chapter 9 of [30]) put operation details within a loop. In the loop body, it tests synchronization conflict with other processes. If no conflict is found, the update will proceed; otherwise, it will go back to the start of the loop and retry. Dynamic partial order reduction [25] relies on a stateless search and thus cannot handle systems with loops, while a cartesian one uses a stateful search and can handle loops.

For convenience, we describe the preceding notion of path in a more succinct notation by omitting immediate states, e.g., \((s_0, \alpha_1, \alpha_2, \ldots, \alpha_n, s_n)\). A legal path of process \(P\) is a path that has at least one transition and all its transitions are executed by process \(P\). Given an LTS \(L = (S, \text{init}, \text{Act}, \rightarrow)\), for each \(s \in S\) and \(\alpha \in \text{Act}\), function \(\alpha(s)\) returns the set of \(\alpha\)-successors of \(s\). That is, \(s' \in \alpha(s)\) iff \(s \xrightarrow{\alpha} s'\).

The following defines the notion of actions being independent, which is central for any partial order reduction.

**Definition 7 (Independence).** Given an LTS \(L = (S, \text{init}, \text{Act}, \rightarrow)\) and \(\alpha, \beta \in \text{Act}\) from different processes, \(\alpha\) and \(\beta\) are independent if for any \(s \in S\) with \(\alpha, \beta \in \text{enabled}(s)\); \(\alpha \in \text{enabled}(\alpha(s))\), \(\alpha \in \text{enabled}(\beta(s))\) and \(\alpha(\beta(s)) = \beta(\alpha(s))\). \(\alpha\) and \(\beta\) are dependent if \(\alpha\) and \(\beta\) are not independent.

For instance, in Algorithm 3.2, any pair of statements of concurrent read operations is independent. Their ordering does not influence the execution result. In our setting, we define that two actions are dependent if two actions access the same variable, and at least one action writes the variable.

The standard semantics of a concurrent program can be regarded as controlled by a special scheduler. The scheduler nondeterministically picks one process to be executed after each transition. Cartesian semantics is proposed as a new operational semantics in cartesian partial order reduction [27] in order to bypass many unnecessary context switches and meanwhile to preserve soundness and completeness. It is based on the notion of cartesian vectors, which identifies for a state a sequence of actions that each process can perform without context switches from that state. The intuition behind cartesian semantics is when cartesian semantics starts the execution from a state \(s\), it selects a sequence of actions \(es\) for each process which are all independent from other process except for the last, and executes them. When the process reaches the target state of the last action in \(es\), it starts the procedure again from this state.

**Definition 8 (Cartesian Vector).** In a concurrent system with \(N\) processes \(P_1, P_2, \ldots, P_N\), a vector \((p_1, \ldots, p_N) \in \text{Path}^N\) is a cartesian vector from a state \(s\) if for each two processes \(P_i, P_j\) such that \(i \neq j\) the following holds:

- The first state of \(p_i\) is \(s\);
- \(p_i\) is a legal path of process \(P_i\);
- \(\forall t \in p_i, t' \in p_j : \text{actions } t \text{ and } t' \text{ are dependent } \implies t \text{ and } t' \text{ are the last actions of } p_i \text{ and } p_j \text{ respectively}.

Since visible actions affect specification states during refinement checking, here we require that partial order reduction is only used for invisible actions in the implementation model. That is, for a cartesian vector \((p_1, \ldots, p_N)\), if a visible action \(\alpha\) exists in any \(p_i\) such that \(1 \leq i \leq N\), then \(\alpha\) must be the last action in \(p_i\). Thus, we consider a slightly modified version of the cartesian function \(\phi : \text{Path} \rightarrow \text{Path}^N\) to generate a cartesian vector for each visited state in [27], presented as Algorithm A.1 in Appendix A. The cartesian semantics generated by \(\phi\) is formalized as a binary relation \(\rightarrow_{\phi}\), which only relates the last states of cartesian vectors and is transitively closed.

An important property of cartesian semantics for linearizability checking is described in the following theorem, which says that if a state that violates linearizability is found with standard semantics, then this state can be found with cartesian semantics.

**Theorem 6.** For a cartesian function \(\phi\) and a given state that violates linearizability \((q, \emptyset)\), if \(s \xrightarrow{\phi} (q, \emptyset)\), then \(s \xrightarrow{\phi} (q, \emptyset)\).

**Proof:**

Given a state that violates linearizability \((q, \emptyset)\), there exists a path \(\pi\) of the form \((s = s_0, \alpha_1, \alpha_2, \ldots, \alpha_n, s_n = (q, \emptyset))\). By a simple argument, we know that \(\alpha_n\) must be a visible action. We shall prove that \(s \xrightarrow{\phi} q\) by induction on \(|\pi|\).

**Basis:** \(|\pi| = 0\). The claim holds.

**Induction hypothesis:** Assume that the claim is true for any path \(\pi\) such that \(|\pi| < n\).

**Induction step:** Consider a path \(\pi\) of the form \((s = s_0, \alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_n, s_n = (q, \emptyset))\).

Let \((p_1, p_2, \ldots, p_k)\) be the cartesian vector that is returned by \(\phi(im)\). Suppose that \(\alpha_n\) is performed by process \(u\) such that \(1 \leq u \leq k\). Because \(\alpha_n\) is visible, it has to be the last action in \(p_u\).

Let \(i\) be the first occurrence of an action in \(\pi\) which ends a path \(p_j\) in \(\phi(im)\), i.e., \(i \in \{1, \ldots, n\}\) is the smallest number for which there exists \(j \in \{1, \ldots, k\}\) such that \((s_0, \alpha_1, \ldots, \alpha_i, s_i)\) contains \(p_j\) transitions of process \(P_j\). Thus, each \(\alpha_m\) is invisible for \(1 \leq m < i\).

We distinguish between the following two cases:
Case 1: \( \alpha_i = \alpha_n \). In this case, \( \alpha_n \) is the only action which ends a path in \( \phi(\text{im}) \), for any \( s \in \pi \). From Lemma 2 in [27], we conclude that every two actions of different processes in \( \langle s_1, \alpha_1, \ldots, \alpha_i, s_1 \rangle \) are independent. Therefore by successively permuting pairs of adjacent independent transitions we can convert \( \pi \) to a new path \( s = s_0, \alpha_1, \alpha_2, \ldots, \alpha_n, s_n' = (\text{im}', \text{sp}') \). From Corollary 1 in [27], \( \text{im}' = q \). As all actions except \( \alpha_n \) are invisible, \( \text{sp}' = \emptyset \) and thus \( s_n' = s_n' \). So \( s \rightarrow_{\phi} (q, \emptyset) \).

Case 2: \( \alpha_i \neq \alpha_n \). From Lemma 2 in [27], we conclude that every two actions of different processes in \( \langle s_0, \alpha_1, \ldots, \alpha_i, s_i \rangle \) are independent. Therefore by successively permuting pairs of adjacent independent transitions we can convert \( \pi \) to a new path \( \langle s = s_0, \alpha_1, \alpha_2, \ldots, \alpha_n, s_n' = (\text{im}', \text{sp}') \rangle \) that begins with \( |p_j| \) actions of \( P_j \). From Corollary 1 in [27], \( \text{im}' = q \). Since every permutated action is invisible, \( s_n' = s_n \). According to the definition of \( \rightarrow_{\phi} \), \( s_0 \rightarrow_{\phi} s_{|p_j|}' \). From the induction hypothesis, we get \( s_{|p_j|}' \rightarrow_{\phi} s_n \). From pseudo-transitivity in [27], \( s_0 \rightarrow_{\phi} s_n \), i.e., \( s \rightarrow_{\phi} (q, \emptyset) \).

Furthermore, if a state that violates linearizability is reached by a path in cartesian semantics, it is straightforward to show that this path also exists in standard semantics. Therefore, the following theorem can be established to guarantee the correctness of applying cartesian partial order reduction to linearizability checking.

**Theorem 7.** A linearizability violated state is reachable in standard semantics if and only if it is also reachable in cartesian semantics.

Algorithm 4.4 is an on-the-fly linearizability checking algorithm with cartesian partial order reduction. The underlined text shows the differences compared with Algorithm 4.1. Given \( X \in S_d \) and an action \( \alpha \), the function \( \text{exec}(X, \alpha) \) returns the successor state of executing \( \alpha \) from \( X \), i.e., \( \text{exec}(X, \alpha) = X' \) such that \( X \xrightarrow{\alpha} X' \). Given a path \( \pi = \langle s = s_0, \alpha_1, \alpha_2, \ldots, \alpha_n, s_n \rangle \) of a cartesian vector \( \phi(s) \), we use \( \text{lastAction}(\pi) \) to denote \( \alpha_n \) and \( \text{lastState}(\pi) \) for \( s_n \). In order to prevent \( \phi(s) \) from generating infinite paths, \( \phi(s) \) stops extending a path once a loop has been detected and marks such path as \( \text{infinite} \). An \( \text{infinite} \) path from \( s \) can only contain invisible actions, since any visible action ends the path as required. Therefore this path is removed from linearizability checking.

As shown above, the key step of this approach is to calculate the cartesian vector for each visited state. So we describe an execution of cartesian vector of \( \phi \) from the initial state of the concurrent stack example in the following steps. We refer to the two sequences of transitions found for a state as a cartesian vector for that state. The transitions enclosed in braces are executed atomically.

At the beginning, because the first actions of both processes are invocation actions, which are visible, each sequence of the cartesian vector only contains its invocation action.

After finding the two sequences, we nondeterministically pick one of them. For example, suppose we first execute \( \text{push}_\text{inv}.1 \). At the resultant state \( (s_3, t_3) \), the cartesian vector is:

\[
\begin{align*}
&\text{p}_1 : \text{push}_\text{inv}.0; \\
&\text{p}_2 : \text{ss} := H; \{H := H + 1; \text{ss} := H;\}; \text{push}_\text{res}.1.1;
\end{align*}
\]

Again, we nondeterministically pick one of these sequences and execute it entirely without a context switch. Suppose we choose to execute \( \text{push}_\text{inv}.0 \) and get to the state \( (s_4, t_4) \). The cartesian vector is:

\[
\begin{align*}
&\text{p}_1 : \text{ss} := H; \{H := H + 1; \text{ss} := H;\}; \\
&\text{p}_2 : \text{ss} := H;
\end{align*}
\]

Notice that the statement in \( \text{p}_2 \) is dependent on the statement \( \{H := H + 1; \text{ss} := H;\} \) in \( \text{p}_1 \).

If we let process 1 first execute its sequence from \( (s_4, t_4) \), then at the resultant state \( (s_9, t_4) \), the cartesian vector is:

\[
\begin{align*}
&\text{p}_1 : \text{push}_\text{res}.0.1; \\
&\text{p}_2 : \text{ss} := H; \{H := H + 1; \text{ss} := H;\}; \text{push}_\text{res}.1.2
\end{align*}
\]

Hence, in this case, the cartesian semantics save 9 transitions and avoids storing 13 states indicated by dashed line, as illustrated in Fig. 3.

**4.4 Combining Symmetry Reduction and Partial Order Reduction**

The combination of symmetry reduction and partial order reduction was first studied by Emerson et al. [22]. They proposed an abstract framework for combining these two reduction techniques based on both preserving simulation relations, and provided model checking algorithms for LTL-X and CTL*-X.
formula with simultaneous symmetry and partial order reductions. Iosif [33] later adopted the above algorithms for dynamic programs, in which processes and objects are created and destroyed with their ongoing executions.

In this subsection, we present an on-the-fly linearizability checking algorithm (presented in Algorithm 4.5) that combines symmetry reduction and cartesian partial order reduction simultaneously. There are two main reasons why we design our own approach to combining symmetry reduction and partial order reduction. One is due to the difference between classical temporal logic and refinement model checking. Their checking algorithms are different. Further, the expressive power of LTL and refinement is different. The relationship between refinement checking and LTL model checking has been studied before [36], [39]. On one hand, refinement can specify properties which cannot be specified using LTL, like "an a happens in every other state" [62]. On the other hand, any LTL property can be captured by refinement checking. In [Leuschel2001], Leuschel et al. proposed an translation from LTL to CSP processes via Bucci automata with some special treatment. The downside of this approach is discussed in [44], “this approach is not that useful in practice (because the expressiveness is on the wrong side of the refinement check for FDR to be efficient, and because it requires several tools to be applied in sequence)”. Thus, two methods are not interchangeable; one cannot replace the other. The other reason is that the partial order reduction approach applied in our setting is different from the two related works. We use cartesian partial order reduction, which [22] and [33] use static approaches based on ample sets. We need to find a cartesian function that works on the symmetry-reduced LTS instead of an ample function for each state [9]. Due to this diversity, the previous combination approaches do not suit it well for our refinement checking.

The correctness of Algorithm 4.5 is established in the following theorem.

Theorem 8. Algorithm 4.1 finds a linearizability violated state (q, ∅) if and only if Algorithm 4.5 finds (Rep(q), ∅).

Proof:

Necessary condition: Suppose Algorithm 4.1 finds a path π that reaches s = (q, ∅). If |π| = 0, it is trivial that the claim holds. Otherwise, given a path π of length n of the form (s0 = (initim, initsp), α1, s1, α2, · · · , sn = (q, ∅)), we shall prove that there exists a pathπ generated by Algorithm 4.5 that reaches (Rep(q), ∅).

By Theorem 6, there exists a path πc of the form (s0′ = (initim, initsp), α′1, s′1 = (im, sp), α′2, · · · , s′n = (q, ∅)) with cartesian semantics (but not omitting the intermediate actions and states of legal paths) that reaches s. By Theorem 4, there exists a repPair-twisted path πr of the form (s0″ = (Rep(initim), Rep(initsp)), α′1, γ1, (Rep(im1), γ1(sp1)), γ1(α′2), · · · , s′n = (Rep(q), ∅)). For any state s′k of πc that is the first state of some legal path (s′k, α′k+1, s′k+1, · · · , α′k+t, s′k+t), for state γ(s′k) where γ ∈ G, its legal path is γ(s′k), γ′(s′k+1), · · · , γ′(s′k+t). Then we can create a path π of the form (s0, α′1, γ1, · · · , γn) from πc and πr in the following way: for all 0 ≤ i ≤ n, if s′i = (im, sp) is the first state of some legal path (s′i, α′i+1, · · · , α′i+t, s′i+t), (in this case s′i = (Rep(im), γ1(ηi(mi), sp)), where η = γi−1 · · · γ1, for all 0 < m < t, s′i+m = η(s′i+m), α′i+m+t = η(α′i+m+t), s′i = s′n and s′i+t = s′i+t). So π is a path generated from Algorithm 4.5. Since s′n must be the last state of some legal path, s′n = s′n. Hence, Algorithm 4.5 finds (Rep(q), ∅) via π.

Sufficient condition: Suppose Algorithm 4.5 finds a path π that reaches (Rep(q), ∅). If |π| = 0, it is trivial that the claim holds. Otherwise, suppose that π = (s0, α′1, γ1, · · · , s′n) where n > 0. Any state in π must be either the first state of some legal path, or any state between the first and the last ones exclusively in the path with cartesian semantics. Pick any state s′i that is the first state of some legal path s′i = (s′i, α′i+1, · · · , α′i+t, s′i+t), by Lemma 2, there exists a repPair-twisted path πi = (s0, α1, γ1, · · · , γi, si, αi+1, · · · , αi+t, si+t) where for all 0 ≤ m ≤ t : si+m = repPair(s′i+m) and for all 0 < k ≤ t : αi+k = repPair(α′i+k). Because s′i and si+t are the first and last states of πi, respectively, s′i = repPair(s′i) = si and si+t = repPair(si+t) = si+t. Then we replace πi by (s′i, αi+1, si+1, · · · , si+t). We continue to replace each legal path in π in this way and get a repPair-

Algorithm 4.5 Linearizability checking algorithm with symmetry reduction and partial order reduction

Procedure linearizability_both (Lim, Lsp)
1: checked := ∅;
2: pending.push((initim, τ(τ(initsp))));
3: while pending ≠ ∅ do
4: (sm, sp) := pending.pop();
5: checked := checked ∪ {(sm, sp)};
6: if sp = ∅ then
7: return false
8: end if
9: for all p ∈ φ(im) do
10: if p is not marked as infinite then
11: im′ = lastState(p);
12: (repIm′, γ0, γ1, · · · , γn) = Rep(im′);
13: sp′ = exec(sp, lastAction(p));
14: if ∀0 ≤ i ≤ n : (repIm′, γi(sp′)) ∉ checked
15: pending.push((repIm′, γ0(sp′)));
16: end if
17: end if
18: end for
19: end while
20: return true
**twisted** path that ends at \((\text{Rep}(q), \emptyset)\). Therefore, by Theorem 4, \(L_{\text{im}} \supseteq_T D_{sp}\) has a path that ends at \((q, \emptyset)\). So the claim holds.

### 5 Experiments

We have implemented our method in the PAT model checker [51] and applied it to a number of concurrent algorithms, including register—the K-valued register algorithm\(^{12}\) in Section 3, counter—the concurrent counter algorithm presented in Example 2, queue—a concurrent non-blocking queue algorithm in Fig. 3 of [42], buggy queue—an incorrect queue algorithm [47] and SNZI—the first algorithm for scalable non-zero indicators [20]. The processes accessing the concurrent data structures are modeled as calling all possible operations nondeterministically. Table 1 summarizes part of our experiments, where ‘-’ means our implementation ran out of memory and ‘(points)’ means that linearization points are given and ‘Gain’ means the relative improvement on the number of states and time consumed brought by the combination of symmetry reduction and partial order reduction. All relevant experiment information is available online [1].

From Table 1, we can see that the number of states and running time increase rapidly with data size and the number of processes. This is not surprising because model checking linearizability is in \(\text{EXPSPACE}\) for both time and space [2]. When linearization points are known, the complexity is still \(\text{EXPSPACE}\), but the state space reduces significantly since the state spaces of implementation and specification are smaller. This is reflected from the stack examples with linearization points in Table 1.

From Table 1, we can see that the number of states and running time increase rapidly with data size and the number of processes. The results conform to theoretical results [2]: model checking linearizability is in \(\text{EXPSPACE}\) for both time and space. When linearization points are known, the complexity is still \(\text{EXPSPACE}\), but the state space reduces significantly since the state spaces of implementation and specification are smaller. This is reflected from the counter examples with linearization points in Table 1. The consumed memory and time for the 4-valued register algorithm are plotted in Figure 4, those for the counter algorithm of size 4 are in Figure 5 and those for the counter algorithm of size 4 with given linearization points are in Figure 6. For the case of the same algorithm and the same number of processes, data are not available for some checking algorithms, as the memory consumptions for running them were beyond the limit of our server.

As can be seen from the figures, compared with the original algorithm, the rates of increase for used memory by the symmetry reduction, partial order reduction and their combination algorithms decrease significantly. Symmetry reduction outperforms partial order reduction in line with the increasing number of processes. The combining has the lowest rate of memory increase. Further it can be seen from the table that the combination reduces the number of states by more than 95% on average. Since the bottleneck in this verification task is the memory consumption, the significant improvement brought by the combination postpones the manifestation of the state space explosion problem till deeper levels and therefore allows the verification to complete for some cases that could not complete before, e.g., 7 processes for 4-valued register and 6 processes for counter of size 7.

On the other hand, the improvement on used time of symmetry reduction, partial order reduction and their combination is not as desirable as that on used memory. Symmetry reduction sometimes improves the running time and sometimes not. So does partial order reduction. The computational overhead of symmetry reduction stems from checking whether the representative state of the orbit of a visited state has been explored. For each state, we generate all of its automorphisms and pick the lexicographically smallest state as the canonical representative. Thus calculating canonical representative states is costly if there are a large number of automorphisms. Take the counter algorithm with linearization points of size 4 as an example. The 7-process case has 7 times the number of automorphisms of the 6-process case, which slows down the linearizability checking a lot. The overhead of partial order reduction is due to dependency analysis between transitions of different processes at each exploration step. Although symmetry reduction (or partial order reduction) by itself does not always improve running time, their combination provides additional improvement, and thus the overall overhead is well compensated by the time we save by combining the two reductions in most cases.

As a result, the combination of both techniques works better than both of them applied in isolation for most cases. This reflects the fact that symmetry and partial order reductions are two orthogonal strategies and can complement each other. The experiments show that our optimization approach can significantly save time and space for demonstrating absence of errors and at the same time it does not sacrifice the capability of detecting bugs.

When the linearizability checking fails, a counterexample trace is returned. In the buggy queue example, after analyzing the counterexample trace, it suggests that the dequeued data item is not the first one in the queue, which violates the sequential specification of the queue object.

Experiments suggest that PAT is faster than FDR for systems without variables [49], [61]. Modeling variables using processes and lack of partial order

\(^{12}\) We extend this example with multiple \(k\) readers and a single writer. The correctness is verified using PAT.
Fig. 4. Performance comparison for the 4-valued register algorithm

Fig. 5. Performance comparison for the stack algorithm of size 4

Fig. 6. Performance comparison for the stack algorithm of size 4 with given linearization points
Using theorem provers

Much work has been done on proofs using theorem provers [19], [10], [11], [13], [12]. In these works, Input/Output automata (IOA) are used to model (correct) abstract data structures and concrete implementation algorithms. Linearizability is proved by showing a simulation relation between the abstract automata and implementation automata. The simulation relation is defined in two parts: an abstraction relation relating the abstract and concrete object values, and a step correspondence relating the abstract and concrete program counter values. The proofs have been mechanised using the PVS theorem prover and a number of theories that embody IOA definitions.

Derrick, Schellhorn and Wehrheim describe a modular approach to establishing linearizability in [15], [14], [16]. Their approach has two parts. First, a generic theory is introduced that encodes linearizability as a special case of data refinement. Local proof obligations for each process are derived based on the theory, and mechanically checked via the KIV theorem prover to make sure that they are sufficient to guarantee linearizability. Second, in practice, a forward simulation relation is built between the concrete and abstract implementations in order to prove data refinement. How to construct algorithm-specific simulation conditions is demonstrated through the lock-free stack algorithm taken from [10] and the lock-coupling list-based set algorithm taken from [56]. Still, this approach requires that linearization points and symmetry reductions will make FDR even slower. Therefore we skip comparison with FDR on these examples.

6 RELATED WORK

Formal verification of linearizability has been studied extensively, since linearizability is a central property for the correctness of concurrent algorithms. We discuss and compare with previous approaches in the literature.

Manual proofs Herlihy and Wing [31] coin the notion of linearizability and present a methodology for verifying linearizability by defining a function that maps every state of a concurrent object to the set of all possible abstract values representing it. Vafeiadis et al. [56] show how to apply a rely-guarantee reasoning approach to verifying linearizability for a family of linked list implementations of a set that employ various fine-grained synchronization techniques. Neither approach requires statically determined linearization points, but these manual proofs typically involve a long and repetitive process and require strong expertise on the specific algorithms. Further, there is a great possibility of making subtle mistakes, which are difficult to identify.

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be statically identified. There are known algorithms which do not satisfy this requirement. This motivates their very recent work [17], which handles the case that a concrete operation mapped to an abstract read-only operation may have linearization points outside the process executing it.

However, theorem prover based approaches are not fully automatic, e.g., conversion to IOA and use of a theorem prover like PVS require strong expertise. Moreover, they constrain the positions of linearization points and thus cannot be applied to all cases. Therefore, an often cited drawback of theorem provers is that they require a great amount of human effort, which hinders their widespread adoption and usage. On the positive side, such tools can reason about infinite state spaces and complicated data structures in a much more effective way than model checking, such that they can guarantee correctness of an algorithm in all possible scenarios.

Static Analysis Amit et al. [3] present a static analysis for verifying linearizability of concurrent unbounded linked data structures using a shape difference abstraction that tracks the difference between two heaps. This approach simultaneously analyzes the concurrent implementation and an executable sequential specification (i.e., a sequential implementation). The two implementations manipulate two disjoint instances of the data structure. The analysis maintains a partial isomorphism between the memory layouts of the two instances. There are several limitations for this approach: (a) every concurrent operation has a (specified) fixed linearization point; (b) the approach verifies linearizability for a fixed but arbitrary number of threads; (c) the approach assumes a garbage collected environment; (d) this approach works well, if the concrete heap and the abstract heap have almost identical shapes. Later, Manevich et al. [41] improve the shape analysis above to handle a larger number of threads. The key idea is to abstract the global heap by decomposing it into (not necessarily disjoint) subheaps, abstracting away some correlations between the subheaps. Decomposition allows reusing subheaps that were decomposed from different heaps, thus representing a set of heaps more compactly (and more abstractly). The resultant algorithm is exponentially faster than the one in [3], being polynomial in the number of threads. Our initial empirical results confirm that our algorithm is able to prove linearizability with 20 threads, ten times more than in [3]. Recently, Berdine et al. [6] further extend this direction to handle an unbounded number of threads. Their algorithms are based on a new abstract domain whose elements represent thread-quantified invariants, i.e., invariants satisfied by all threads. We exploit existing abstractions to represent the invariants. As a result, our technique lifts existing abstractions by wrapping universal quantification around elements of the base abstract domain.

Vafeiadis [54] further improves this solution to allow linearization points in different threads. Their main limitation, like manual approaches, is that users need to provide linearization points, which are unknown for some algorithms.

This motivates Vafeiadis’s work [55] on automating the entire verification process. That work defines the execution of a concrete operation, which maps to the execution of an abstract read-only operation, as a pure execution. It assumes that the linearization points with intrinsic conditions or residing within other processes can appear only in a pure execution, similar to [17]. For each pure execution, a linearizability checker is instrumented into each program point of all processes; for other executions, a checker is instrumented to only monitor the statements of the currently executing process. Then an abstract interpreter is used to check linearizability violations.

Model checking Vechev and Yahav [58] provide two methods for linearizability verification using model checking techniques.

The first method explores, for every history, all possible linearizations, trying to find one that satisfies the sequential specification. Hence, its worst-case time is exponential in the length of the history, as it may have to try all possible permutations of the history. As a result, the number of operations they can check is only 2 or 3. In contrast, our approach handles all possible interleaving of operations given sizes of the shared objects. Because of partial order reduction and symmetry reduction, our approach is more scalable than theirs. The second method requires algorithm-specific user annotations for linearization points, which makes it easy to check whether it satisfies the sequential specification. Hence this method scales better than the first one. However, it is not generic because not all algorithms have explicit linearization points.

Burckhardt et al. [7] present an automatic linearizability checker Line-Up based on the stateless model checker CHESS. Given a deterministic sequential specification, Line-Up is complete but only sound with respect to given inputs. Meanwhile, they generalize the notion of linearizability to handle blocked execution histories.

A new technique designed for concurrent linked-list implementations has been proposed by Cerny et al. [8]. A common pattern in these implementations is that a list entry has a data value from an infinite domain equipped only with the equality and order testing. This pattern makes it possible to represent list content as a data word in automata theory. Cerny et al. reduce the problem of verifying linearizability to the reachability problem of method automata which simulate how the operations manipulate a concurrent object. They prove that linearizability is decidable for a bounded number of operations. The upside of
their approach is that it allows a concurrent object to be stored in a singly-linked unbounded heap where the element stored in each location comes from an unbounded data domain; the downside is that it is only capable of checking the executions of two fixed operations due to the severe state explosion problem.

As a coin has two sides, model checking approaches have virtues and limitations. They significantly relax the requirement on user expertise and effort. Most of them do not rely on the knowledge of linearization points, nor do they require users to come up with hints to the algorithm in question. The limitation is that the infamous state explosion problem cripples their ability to guarantee the correctness without bounding the data structures and processes in parallel. Our reduction approach clearly does not eliminate the state explosion problem. Yet, it postpones its manifestation till deeper levels.

In terms of modeling of linearizability, our approach is based on the trace refinement of LTSs, which is similar to [2]. Our refinement checking algorithm is related to existing on-the-fly behavioral equivalence and pre-order checking algorithms (e.g., [45], [18]). The non-atomic refinement defined in [14] separates the data explicitly as state-based formalism Object-Z. That modeling requires the knowledge of linearization points, and also prevents automatic verification techniques such as model checking to be used.

7 Conclusion

In this work, we studied the formal verification of linearizability using model checking techniques. The key idea of this work is to construct a linearizable specification and express linearizability as the trace refinement relation between the specification and the target concurrent algorithm. Based on whether the linearization points are known or not, we defined two ways of using refinement relations to express linearizability. Based on these definitions, we developed a model checking algorithm for linearizability verification. To further improve the performance, two reduction techniques were proposed, i.e., dynamic partial order reduction and symmetry reduction. Furthermore, we proved that these two reductions are orthogonal to each other and can be applied together. Experimental results showed that our approach is capable of verifying practical algorithms including two recent algorithms: the mailbox problem and scalable pre-order checking algorithms ([49], [18]).

The main drawback for our approach is that model checking can only work if the target system has a finite number of states, which limits our approach to work with bounded data structure and finite number of threads. To tackle this problem, several directions are possible. Firstly, one can use abstraction techniques to reduce an infinite number of threads to a small number, e.g., process counter abstraction. The challenge here is how to find the right level of abstraction and detect spurious counterexamples. Secondly, shape analysis has been proved to be a successful static analysis technique for verifying linearizability. We plan to look at the possibility of incorporating this technique in our model checking to handle unbounded data size. The main idea is to use shape analysis as a kind of data abstraction method to reduce an unbounded data structure to finite shapes. The challenge is to prove that the abstraction is sound.

APPENDIX A

Cartesian Function

Let $P_1, P_2, \ldots, P_k$ be the processes of the concurrent object algorithms; $\alpha_i$ denotes the action executed by process $P_i$.

Algorithm A.1 Algorithm for calculating cartesian vectors on $L_{im}$

```plaintext
Helper function:
nextTrans(s, i) = (\alpha_i, s') : s \rightarrow s' \in T_{im}
```

Prob procedure $0(s)$

1: for all $s \rightarrow s' \in T_{im}$ do
2: add $\alpha_i$ and $s'$ to $CV[i]$;
3: end for
4: extendable := \{1, \ldots, n\};
5: for all $i \in \{1, \ldots, n\}$ : lastAction($CV[i]$) is visible do
6: extendable := extendable $\rightarrow \{i\}$;
7: end for
8: for all $i, j \in$ extendable : $i \neq j \land$ lastAction($CV[i]$) is dependent on lastAction($CV[j]$) do
9: extendable := extendable $\rightarrow \{i, j\}$;
10: end for
11: while extendable $\neq \emptyset$ do
12: pick any $i \in$ extendable;
13: $s :=$ lastState($CV[i]$);
14: $(\alpha_i, s') :=$ nextTrans($s, i$);
15: if $\exists j \neq i : \alpha_i$ is dependent on some action in $CV[j]$ (other than the last) then
16: extendable := extendable $\rightarrow \{i\}$;
17: else
18: for all $j \neq i : \alpha'_i$ is dependent on lastAction($CV[j]$) do
19: extendable := extendable $\rightarrow \{i, j\}$;
20: end for
21: if $s' \in CV[i] \land i \in$ extendable then
22: mark $CV[i]$ as infinite;
23: extendable := extendable $\rightarrow \{i\}$;
24: end if
25: if $\alpha_i$ is visible $\land i \in$ extendable then
26: extendable := extendable $\rightarrow \{i\}$;
27: end if
28: add $\alpha_i$ and $s'$ to $CV[i]$;
29: end if
30: end while
31: return $CV$
```


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