Beyond Normality: A Distributionally Robust Stochastic User Equilibrium Model *

Selin Damla Ahipasaoglu† Rudabeh Meskarian‡ Thomas L. Magnanti§
Karthik Natarajan¶

June 15, 2014

Abstract

The Stochastic User Equilibrium (SUE) model predicts traffic equilibrium flow assuming that users choose their perceived maximum utility paths (or perceived shortest paths) while accounting for the effects of congestion that arise due to users sharing common links. Inspired by recent work on distributionally robust optimization in choice models, specifically a Cross Moment (CMM) choice model, we develop a new distributionally robust SUE model that uses only mean and covariance information on path utilities. We show that under mild conditions, the CMM-SUE (Cross Moment-Stochastic User Equilibrium) exists and is unique. By combining a simple projected gradient method to evaluate path choice probabilities with a gradient descent method, we show that the CMM-SUE is efficiently computable. CMM-SUE provides both modeling flexibility and computational advantages over approaches such as the well-known MNP-SUE (Multinomial Probit-Stochastic User Equilibrium) model that require distributional (normality) assumptions to model correlation effects from overlapping paths. In particular, it avoids the use of simulation methods employed in computations for the distributionally-based MNP-SUE model. Preliminary computational results indicate that CMM-SUE provides a practical distributionally robust alternative to MNP-SUE.

1 Introduction

Traffic assignment problems have intrigued system planners and researchers for several decades. Their key challenge is to develop reasonable models and methods to predict traffic flow by incorporating phenomenon such as congestion effects and uncertainty in route preferences. One of the important traffic assignment concepts is the notion of the user equilibrium that reflects the aggregate outcome of individual decisions made by drivers who choose routes from origin to destination.

*This project was partly funded by the SUTD-MIT International Design Center grant number IDG31300105 on ‘Optimization for Complex Discrete Choice’ and the MOE Tier 2 grant number MOE2013-T2-2-168 on ‘Distributional Robust Optimization for Consumer Choice in Transportation Systems’.
†Engineering Systems and Design, Singapore University of Technology and Design, 20 Dover Drive, Singapore 138682. Email: ahipasaoglu@sutd.edu.sg
‡Engineering Systems and Design, Singapore University of Technology and Design, 20 Dover Drive, Singapore 138682. Email: meskarian@sutd.edu.sg
§Singapore University of Technology and Design and MIT, Room 32-D784, Cambridge, MA 02139. Email: magnanti@sutd.edu.sg
¶Engineering Systems and Design, Singapore University of Technology and Design, 20 Dover Drive, Singapore 138682. Email: natarajan_karthik@sutd.edu.sg
nodes on the network. The simplest traffic equilibrium is the deterministic Wardropian User Equilibrium [47] which is derived from the following principle:

“At equilibrium, the travel costs on all routes that are actually used are equal and less than those which would be experienced by an user on any unused route”.

Users seek to minimize their own costs of transportation with the traffic flow that results from this principle leading to a User Equilibrium (commonly referred to as UE). At equilibrium, no user can reduce his/her own costs by unilaterally shifting from one route to another. Properties such as the existence of the equilibrium, the uniqueness of the equilibrium and the computation of the equilibrium have been analyzed using techniques from convex optimization [see Beckmann, McGuire and Winston [7]], variational inequalities [see Smith [45] and Dafermos [19]] and nonlinear complementarity problems [see Ashtiani and Magnanti [1]]. In this paper, we focus on the optimization approach and use recent advances in distributionally robust optimization to solve stochastic traffic equilibrium problems.

The Stochastic User Equilibrium (SUE) model introduced by Daganzo and Sheffi [22] generalizes the deterministic UE model by allowing the users to perceive the travel costs possibly differently from the actual realized travel costs. Users choose routes based on their perceived route costs, thus splitting up the aggregate demand for an origin-destination pair among the paths connecting them. From a system planner’s view, the perceived travel costs are not observable and hence modeled as random variables. The probabilities represent the fraction of users who move from the origin to the destination using a particular path. The SUE is derived from the principle that:

“At equilibrium, no driver can improve on his/her perceived travel cost by unilaterally changing routes”.

To model the randomness in the perception of travel costs and its effect on user route choice behavior, discrete choice models have been extensively used in traffic assignment problems. We review the research on discrete choice models in a transportation context most relevant to this paper next. The reader is referred to Prato [40] for a recent comprehensive review on this topic.

1.1 Related Literature on Discrete Choice Models in Transportation

In traffic networks, discrete choice models are used to capture the route choice behavior of travelers. Consider a finite set of alternatives \( K = \{1, \ldots, K\} \) with the perceived utility of an alternative \( k \) modeled as a random variable \( \tilde{u}_k \) consisting of a deterministic component \( \mu_k \) and an additive random error term \( \tilde{\epsilon}_k \):

\[
\tilde{u}_k = \mu_k + \tilde{\epsilon}_k, \quad \forall k \in K.
\]  

Assuming that users are utility maximizers and choose their most preferred alternative, the probability of selecting an alternative \( k \) for a set of users is given by:

\[
p_k = \mathbb{P}_\theta \left( k = \arg \max_{l \in K} (\mu_l + \tilde{\epsilon}_l) \right),
\]

where \( \theta \) is the probability distribution of the vector \( \tilde{\epsilon} \). Discrete choice models vary in the specification of the probability distribution of the random error vector \( \tilde{\epsilon} \) and the corresponding choice probabilities \( p_k \). In traffic networks, the alternatives correspond to the set of routes (paths) for a given origin-destination pair. The deterministic component accounts for the utility of a path as characterized by observable attributes such as travel times, costs and the attractiveness of the route while the random component accounts for other unobservable attributes. In a SUE model, the congestion effect is modeled by letting the deterministic utility of the path to be flow dependent (particularly a decreasing function of the flow value) while the random component is flow independent.
The earliest application of discrete choice models in transportation was developed with the Multinomial Logit (MNL) model (see Dial [24], Fisk [25]). The logit choice probabilities are obtained from a discrete choice model by assuming that the random error terms are independent and identically distributed Gumbel random variables. The choice probabilities under the logit model is given by the following expression:

\[ p_{k}^{\text{mnl}} = \frac{e^{\mu_k}}{\sum_{l \in K} e^{\mu_l}}. \]

One of the advantages of using the logit model in traffic assignment is its analytical formula which make it possible to develop efficient link-based algorithms to compute the MNL-SUE (Multinomial Logit-Stochastic User Equilibrium) [see Dial [24], Fisk [25], Akamatsu [5]]. These methods typically assume that either (a) the choice set is restricted to a set of “efficient” paths, namely all arcs along the path lead the user closer to the destination and further away from the origin [see Dial [24]], (b) the choice set consists of all cyclic (possibly infinite) and acyclic paths [see Bell [10], Akamatsu [4]] or (c) the network is a directed acyclic graph [see Bing-Feng et. al. [15]]. However assumptions (a) and (b) are not always justified behaviorally while assumption (c) is often restrictive for traffic networks [see Bekhor and Toledo [9]]. More importantly in the MNL model the utilities of different routes are uncorrelated which is difficult to justify with overlapping routes. In the MNL model this results in the property of the Independence of Irrelevant Alternatives, which in the route choice context, can be interpreted as a failure to account for similarities between alternatives [see Daganzo and Sheffi [22]].

As an alternative, Daganzo and Sheffi [22] proposed the use of the Multinomial Probit (MNP) model in traffic assignment to capture correlation information. In the MNP model, the error terms are assumed to be multivariate normally distributed with mean 0 and covariance matrix \( \Sigma \succ 0 \). With \( \Sigma \), the probit-based choice model incorporates general correlation structures among the alternatives. The choice probability in MNP models is given by a multidimensional integral of the form:

\[
p_{k}^{\text{mnp}} = \int_{-\infty}^{\epsilon_{1}+\mu_{k}-\mu_{1}} \cdots \int_{-\infty}^{\epsilon_{K}+\mu_{k}-\mu_{K}} (2\pi)^{-K/2} \left| \Sigma \right|^{-1/2} e^{-\frac{1}{2} \epsilon^{T} \Sigma^{-1} \epsilon} \, d\epsilon.
\]

For a large number of alternatives, simulation based methods are used to evaluate the choice probabilities in MNP. Optimization based formulations for the SUE model with choice probabilities from the probit model were developed in the early works of Daganzo and Sheffi [22], Daganzo [21] and Sheffi and Powell [43]. To solve the optimization formulation, the Method of Successive Averages (MSA) [see Powell and Sheffi [38]] in conjunction with Monte-Carlo based simulations was proposed by Sheffi and Powell [43]. In the MSA, step-sizes are chosen in a predetermined sequence to try and avoid finding optimal step-sizes at each iteration of the gradient based algorithm. Convergence of the method was proved in [38, 43]. Maher and Hughes [31] proposed a Stochastic Assignment method using Clark’s approximation for probit choice probabilities to solve the MNP-SUE model approximately. Liu and Meng [29] have recently proposed a distributed computing approach to solve large scale probit based user equilibrium problems. Computation of the MNP-SUE on realistic traffic networks remains a challenge primarily due to the requirement of implementing a large scale simulation based optimization approach. In this paper, we provide a new class of distributionally robust SUE models where the covariance matrix is specified but the assumption of normality is dropped. We provide evidence that this forms a practical alternative to MNP-SUE.

An important issue in the MNP-SUE model is the specification of the covariance matrix \( \Sigma \). While the mean utility vector \( \mu \) is assumed to be flow dependent to model congestion effects, the covariance matrix \( \Sigma \) is assumed to be flow independent. One approach proposed by Daganzo and Sheffi [22] in specifying \( \Sigma \) is to assume that the variance of a link is proportional to a fixed
characteristic of the link (such as the free flow travel cost) while different link error terms are independent of each other. Using the network structure it is then possible to estimate $\Sigma$ for the path utilities from the link utilities. Yai et. al. [10] proposed the use of a structured covariance matrix with an additional path specific variance. In this paper, we allow for a generic specification of the covariance matrix $\Sigma$ of path utilities that could be possibly estimated from link terms and additional path specific terms. Our only requirement is that the matrix $\Sigma$ is strictly positive definite. For a symmetric matrix $A$, we use $A \succ 0$ to denote the matrix is positive definite and $A \succeq 0$ to denote the matrix is positive semidefinite matrix through the rest of the paper.

Generalizations of discrete choice models with logit type choice probability formulas have also been used in transportation applications. This includes the C-logit model [see Cascetta et al. [17]] and the Path-Size logit model [see Ben-Akiva and Bierlaire [13]]. Both these models have choice probability formulas similar to MNL with an additional correction factor. Path-based algorithms for SUE models with both C-logit [see Zhong et. al. [50], Xu et. al. [48]] and Path-Size logit [see Bovy et. al. [16]] have been developed. Other generalization of logit-based choice models using Generalized Extreme Value (GEV) theory have been proposed in the literature. One such example is the Nested Logit model [see Ben-Akiva and Lerman [12]]. A SUE model for Nested Logit has been developed in Bekhor and Prashker [8]. While these models maintain the simplicity of choice probability formulas, none of these models capture general correlation structures as in the MNP model.

It is important to note that an alternate stream of research in traffic equilibrium deals with uncertain travel costs. In this line of work, the travel times are assumed to be random and users select their routes using a risk averse criterion. The equilibrium is generated by the aggregate outcome of decentralized risk averse decisions. Examples include the stochastic Wardrop equilibrium where users select routes by minimizing the sum of the mean and variance of travel costs [see Uchida and Iida [46], Bell and Cassir [11], Nikolova and Stier-Moses [36]], the percentile equilibrium where users select routes by minimizing a percentile of travel cost [see Ordóñez and Stier-Moses [37]] and the robust Wardrop equilibrium where users select routes by solving a robust optimization problem that imposes a limit on the number of arcs that can deviate from the mean [see Bertsimas and Sim [14], Ordóñez and Stier-Moses [37]]. Note that in this class of models the travel costs are random while in SUE models the perceptions of travel costs are random. Our focus in this paper is on SUE models that builds on recent work in distributionally robust optimization in discrete choice models. We review some of the related literature next.

1.2 Related Literature on Distributionally Robust Optimization in Choice

Natarajan, Song and Teo [35] and Mishra, Natarajan and Teo [32] have recently introduced a semiparametric class of discrete choice models using the concept of distributional robustness. In this class of choice models, the joint distribution $\theta$ of the random error vector $\tilde{\epsilon}$ is assumed to be only partly specified. Suppose that $\theta$ is known only to lie in a set of distributions $\Theta$. Consider a distribution $\theta^*$ in the set $\Theta$ that maximizes the expected user utility:

$$\theta^* = \arg \max_{\theta \in \Theta} \mathbb{E}_{\theta} \left( \max_{k \in K} (\mu_k + \tilde{\epsilon}_k) \right).$$  \hspace{1cm} (1.2)

Then, the choice probabilities are evaluated for the extremal distribution $\theta^*$ that attains the maximum expected user utility:

$$p_k = \mathbb{P}_{\theta^*} \left( k = \arg \max_{i \in K} (\mu_i + \tilde{\epsilon}_i) \right).$$
Natarajan, Song and Teo \cite{35} defined $\Theta$ as the set of all possible joint distributions with fixed marginal distributions. The choice probabilities are then evaluated for the distribution with the given marginals that maximizes expected user utility. Their model is referred to as the Marginal Distribution model. Mishra et. al. \cite{33} have recently shown that Marginal Distribution model has interesting connections with the classical MNL model. Particularly in the special case of exponential marginal distributions it exactly generates the logit choice probability. Thus it recreates the well known choice probability formula despite the difference in assumptions on the underlying probability distribution. Using choice data on customer preferences of the safety features of vehicles, they have provided evidence that the Marginal Distribution model provides a practical alternative to existing choice models. However in the Marginal Distribution model, the covariance matrix among the utilities is not explicitly modeled.

An alternative discrete choice model that is of particular relevance to this paper is the Cross Moment (CMM) model proposed by Mishra, Natarajan and Teo \cite{32} which explicitly captures the covariance information among utilities. In the CMM model, the available information on the joint distribution $\theta$ of $\tilde{\epsilon}$ is the first two moments. Let $\tilde{\epsilon} \sim (0, \Sigma)$ denote the set of probability distributions with mean $E_{\theta}[\tilde{\epsilon}] = 0$ and covariance matrix $Cov_{\theta}[\tilde{\epsilon}] = \Sigma$. In the CMM model, the modeler is assumed to solve the following optimization problem:

$$(CMM) \quad Z^* = \max_{\tilde{\epsilon} \sim (0, \Sigma)} E_{\theta} \left( \max_{k \in K} (\mu_k + \tilde{\epsilon}_k) \right). \quad (1.3)$$

The outer maximization problem in (1.3) is over all joint distributions of the random vector that is consistent with the two moment information while the inner maximization problem is the choice problem of an individual user. Problem (1.3) is equivalent to finding a joint distribution for the random vector that maximizes the expected agent utility subject to first two moment information. Mishra, Natarajan and Teo \cite{32} have developed a convex semidefinite optimization approach to find the maximal expected user utility $Z^*$ and the corresponding choice probabilities for the distribution that attains the bound. Computationally solving large scale semidefinite programs is however a challenge. Ahipasaoglu, Li and Natarajan \cite{3} have recently developed an alternate representative agent formulation for the CMM model. In the representative agent model, the aggregate behavior of a set of users is described through the choices made by a single representative user who has a preference for diversity [see Anderson, Palma and Thisse \cite{6}]. Consider a representative agent who chooses a probability vector in the unit simplex defined as:

$$\Delta_K = \left\{ p \in \mathbb{R}_+^K \mid e^T p = 1 \right\},$$

where $e$ is a vector of all ones. Ahipasaoglu, Li and Natarajan \cite{3} have shown that (CMM) is equivalent to solving a nonlinear concave maximization problem over a unit simplex. The main result from \cite{3} is provided next.

**Proposition 1.1 (Ahipasaoglu, Li and Natarajan \cite{3})** Assume that $\Sigma \succ 0$.

(i) The maximum expected user utility $Z^*$ in the CMM model is the optimal objective value of the following nonlinear (strictly) concave maximization problem over the unit simplex:

$$(CMM) \quad Z^* = \max \left\{ \mu^T p + \text{trace} \left( \left( \Sigma^{1/2} S(p) \Sigma^{1/2} \right)^{1/2} \right) \mid p \in \Delta \right\}, \quad (1.4)$$

where $S(p) = \text{Diag}(p) - pp^T \succeq 0$ with $\text{Diag}(p)$ denoting a diagonal matrix with the components of $p$ along the diagonal and $B = A^{1/2}$ denotes the unique positive semidefinite square root of a matrix $A \succeq 0$. 

5
The optimal decision vector $p^*$ in (1.4) is unique and lies strictly in the interior of the simplex. Furthermore, $p^*$ is the choice probability vector for a distribution $\theta^*_{cmm}$ that maximizes the expected agent utility, namely:

$$p^*_k = \mathbb{P}_{\theta^*_{cmm}}(k = \arg \max_{l \in K}(\mu_l + \tilde{\epsilon}_l)).$$

Note that since the multivariate normal distribution is a feasible distribution in formulation (1.3), $Z^*$ provides an upper bound on the expected user utility in MNP. The choice probabilities are however computed in a very different manner in the two models. In the MNP model, numerical integration techniques are used when the number of alternatives is small and simulation techniques are used when the number of alternative is large. On the other hand, in the CMM model, Ahipasaoglu, Li and Natarajan [3] proposed the use of a gradient based optimization technique to compute the choice probabilities. Define:

$$g(p) = -\frac{1}{2} \left( \text{diag} \left( \Sigma^{1/2}(T^{1/2}(p))^{\dagger}\Sigma^{1/2} \right) - 2\Sigma^{1/2}(T^{1/2}(p))^{\dagger}\Sigma^{1/2}p \right),$$

where $\text{diag}(\cdot)$ is the column vector formed by the diagonal elements of the matrix, $A^{\dagger}$ denotes the Moore-Penrose pseudoinverse of matrix $A$ and $T(p) = \Sigma^{1/2}S(p)\Sigma^{1/2}$. The optimality condition for (1.4) is then given by the following conditions:

$$\nabla h(p) := \left( \mu - \frac{e^T\mu e}{K} \right) - \left( g(p^*) - \frac{e^Tg(p^*)e}{K} \right) = 0 \quad \text{and} \quad p^* \in \Delta.$$

Ahipasaoglu, Li and Natarajan [3] have developed a simple projected gradient method, given below, to solve (1.4).

**Input:** Parameters: $\mu$, $\Sigma$, Starting point: $p^0 \in \text{int}(\Delta_{n-1})$, Step length: $\alpha$, Tolerance: $\epsilon$.

**Output:** Solution $p^k$.

1. $i \leftarrow 0$.
2. **for** the stopping criterion is violated, e.g. $\|\nabla h(p^i)\|_2 > \epsilon$ **do**
   1. $\bar{\alpha} \leftarrow \alpha$
   2. $p^{i+1} \leftarrow p^i + \bar{\alpha} \cdot \nabla h(p^i)$
   3. **for** $\min_j p^{i+1}_j \leq 0$ **do**
      1. $\bar{\alpha} \leftarrow \bar{\alpha}/2$
      2. $p^{i+1} \leftarrow p^i + \bar{\alpha} \cdot \nabla h(p^i)$
   **end**
3. $i \leftarrow i + 1$
**end**

**Algorithm 1:** CMM($\mu$, $\Sigma$)

Our focus in this paper is on developing a SUE model based on the CMM choice model. We will later observe that having an efficient algorithm to calculate the CMM choice probabilities will play an important part in calculating the equilibrium arc flows in our new model.

In Section 2, we develop a new stochastic user equilibrium model, referred to as CMM-SUE, using the Cross Moment discrete choice model. We show that under mild conditions, the equilibrium exists and is unique. In Section 3, we develop an algorithm to compute the CMM-SUE using the Method of Successive Averages and a projected gradient method to compute choice probabilities. In Section 4, we present numerical tests to compare the MNP-SUE and CMM-SUE. Concluding remarks are provided in Section 5.
2.1 Preliminaries

A traffic network is represented as a directed graph $G = (N, A)$, with a set $N$ of nodes (for example intersections) and a set $A$ of arcs (for example roads). Let $N$ and $A$ denote the total number of nodes and arcs in the network. Associated with the network is a set of fixed origin-destination pairs indexed by $w \in W$ where $(r_w, s_w)$ denotes the $w$th origin-destination pair. Let $K_w$ denote a finite set of directed simple paths connecting the origin $r_w$ with the destination $s_w$. Our specification of the choice set $K_w$ is flexible in that it is possible to restrict attention to only a subset of paths connecting $r_w$ and $s_w$ that is justified behaviorally. The reader is referred to Prato [40] for a review of various choice set generation methods developed for traffic networks. The set of all paths is denoted by $K = \bigcup_{w \in W} K_w$. Let $d_w > 0$ be the demand associated with the origin-destination pair $(r_w, s_w)$. Each path $k \in K$ is associated with a path flow $x_k$ where $x_k = (x_k)_{k \in K}$. A feasible flow on the network is a set of path flow values $x_k$ satisfying the constraints:

$$\sum_{k \in K_w} x_k = d_w \quad \forall w \in W$$

$$x_k \geq 0 \quad \forall k \in K.$$  

(2.7)

The flow along an arc $a \in A$ is given by the sum of flows on all the paths that contain arc $a$:

$$f_a = \sum_{w \in W} \sum_{k \in K_w : k \ni a} x_k.$$  

Associated with the arc $a \in A$ is a deterministic link cost given by $c_a(f_a)$ that models the effect of congestion on costs. The cost function is assumed to be a nonnegative, strictly increasing, continuous function in the flow value. For a given arc flow vector $f = (f_a)_{a \in A}$, the deterministic cost associated with the path $k$ is given by:

$$c_k(f) = \sum_{a \in A : k \ni a} c_a(f_a).$$  

(2.8)

With slight abuse of notation, we use the terminology $c_k(f)$ and $c_k(x)$ interchangeably.

In the SUE model, the underlying assumption is that drivers make routing decisions using their perceived travel utilities. The perceived utility along route $k$ is modeled as a random variable:

$$\tilde{u}_k(f) = -c_k(f) + \tilde{\epsilon}_k, \quad \forall k \in K$$

where $c_k(f)$ is the travel cost in $(2.8)$ along a route $k$ which models the congestion effects in the links that form the route. The random error term $\tilde{\epsilon}_k$ captures the randomness in the perception of utility associated with path $k$. Note that $\tilde{\epsilon}_k$ is assumed to be flow independent. Given realizations of perceived route utilities, each driver selects the route with maximum utility. The probability that a route $k \in K_w$ is chosen as the most preferred one is flow dependent and given by:

$$p_k(f) = \mathbb{P}_\theta \left( k = \arg \max_{l \in K_w} (-c_l(f) + \tilde{\epsilon}_l) \right).$$

This brings us to the formal definition of the MNP-SUE.

**Definition 2.1 (MNP-SUE)** In the MNP-SUE, the equilibrium arc flow is the solution to the fixed point equation:

$$f_a = \sum_{w \in W} d_w \sum_{k \in K_w : k \ni a} \mathbb{P}_{\theta_{mnp}} \left( k = \arg \max_{l \in K_w} (-c_l(f) + \tilde{\epsilon}_l) \right) \quad \forall a \in A.$$  

(2.9)
where $\tilde{\epsilon}_w = (\tilde{\epsilon}_k)_{k \in K_w}$ is a multivariate normal random vector with mean $0$ and covariance matrix $\Sigma_w$.

Under the assumption that the covariance matrix $\Sigma_w$ is strictly positive definite for each $w$ and the link travel costs are differentiable, positive and strictly increasing in its argument, the equilibrium is known to exist and is strictly unique. Furthermore, unlike the deterministic user equilibrium, the path flow vector is also unique and given by:

$$x_k = d_w p_{\theta_{mn}} \left( k = \arg \max_{l \in K_w} (-c_l(f) + \tilde{\epsilon}_l) \right) \quad \forall k \in K_w, w \in W,$$

(2.10)

Equivalent optimization formulations for the SUE were developed in Sheffi and Powell [43] and Daganzo [21]. In terms of the arc flow decision variables, the unconstrained optimization formulation in Sheffi and Powell [43] is given as follows:

$$\min_f \sum_{w \in W} d_w \mathbb{E}_\theta \left( \max_{k \in K_w} (-c_k(f) + \tilde{\epsilon}_k) \right) + \sum_{a \in A} f_a c_a(f_a) - \sum_{a \in A} \int_0^{f_a} c_a(t) dt,$$

(2.11)

Under the assumptions on the covariance matrix and the cost functions for the MNP choice model, the optimal solution to (2.11) is equivalent to the equilibrium link flows in the MNP-SUE. Though non-convex in general, the first order optimality conditions of (2.11) provides the unique equilibrium link travel cost $c_a(f_a)$ function is assumed to be differentiable, positive and strictly increasing in its argument, the inverse function $f_a(c_a)$ exists, is positive and an increasing function for $c_a > c_a = c_a(0)$. In terms of the arc cost variables $c = (c_a)_{a \in A}$, the strictly convex optimization formulation developed in Daganzo [21] and Sheffi and Powell [43] is as follows:

$$\min_c \sum_{w \in W} d_w \mathbb{E}_\theta \left( \max_{k \in K_w} \left( - \sum_{a \in A: k \geq a} c_a + \tilde{\epsilon}_k \right) \right) + \sum_{a \in A} \int_{\Lambda_a} f_a(c) dc.$$

(2.12)

The optimal solution to (2.12) can be used to find the equilibrium link flows in the SUE model. We build on these optimization formulations by incorporating the concept of distributional robustness to develop a new equilibrium traffic flow.

### 2.2 The New Model

In this section, we develop a new traffic equilibrium concept under the assumption that the mean of the random vector $\tilde{\epsilon}_w$ is $0$ and the covariance matrix is $\Sigma_w$ for $w \in W$. The distributional robust counterpart of (2.11) is formulated by minimizing the maximum objective function as follows:

$$\min_f \sum_{w \in W} d_w \max_{\tilde{\epsilon}_w \sim \theta (0, \Sigma_w)} \mathbb{E}_\theta \left( \max_{k \in K_w} (-c_k(f) + \tilde{\epsilon}_k) \right) + \sum_{a \in A} f_a c_a(f_a) - \sum_{a \in A} \int_0^{f_a} c_a(t) dt,$$

(2.13)

or equivalently the distributional robust counterpart of (2.12) is formulated as:

$$\min_c \sum_{w \in W} d_w \max_{\tilde{\epsilon}_w \sim \theta (0, \Sigma_w)} \mathbb{E}_\theta \left( \max_{k \in K_w} \left( - \sum_{a \in A: k \geq a} c_a + \tilde{\epsilon}_k \right) \right) + \sum_{a \in A} \int_{\Lambda_a} f_a(c) dc,$$

(2.14)

We prove our main result using formulation (2.14). Define for each origin-destination pair $w \in W$, the path choice probability vector $p_w = (p_k)_{k \in K_w} \in \Delta_{K_w}$ with $p = (p_w)_{w \in W} \in \Delta = \Delta_{K_1} \times \ldots \times \Delta_{K_N}$.
\[ \Delta_{KW}. \] Using Proposition [11] to reformulate the inner optimization problem as a maximization problem over the unit simplex, we obtain a reformulation for (2.14) as:

\[
\min_c \sum_{w \in W} d_w \max_{p_w \in \Delta_{KW}} \left( -\sum_{k \in K_w} \left( \sum_{a \in A ; k \neq a} c_a \right) p_k + \text{trace} \left( \left( \Sigma_w^{1/2} S(p_w) \Sigma_w^{1/2} \right)^{1/2} \right) \right) + \sum_{a \in A} \int_{\Delta_a} f_a(c) dc.
\]

Define the function:

\[
\Phi(c, p) = \sum_{w \in W} d_w \left( -\sum_{k \in K_w} \left( \sum_{a \in A ; k \neq a} c_a \right) p_k + \text{trace} \left( \left( \Sigma_w^{1/2} S(p_w) \Sigma_w^{1/2} \right)^{1/2} \right) \right) + \sum_{a \in A} \int_{\Delta_a} f_a(c) dc,
\]

to rewrite the distributional robust optimization problem as:

\[
\min_c \max_{p \in \Delta} \Phi(c, p).
\]

Clearly \( \Phi(c, p) \) is convex with respect to the link cost variables \( c \) and concave with respect to the path choice probabilities \( p \). Since the feasible region for both \( c \) and \( p \) is convex and compact for \( p \) (product of the unit simplices) and the objective function is continuous, classical results in minimax duality hold (see Rockafellar [41], Sion [44]). Define a saddle point \((c^*, p^*)\) of a function \( \Phi \) as:

\[
\Phi(c^*, p) \leq \Phi(c^*, p^*) \leq \Phi(c, p^*) \ \forall c, \forall p \in \Delta.
\]

Then \((c^*, p^*)\) is a saddle point that satisfies the minmax equality:

\[
\min_c \max_{p \in \Delta} \Phi(c, p) = \Phi(c^*, p^*) = \max_{p \in \Delta} \min_c \Phi(c, p).
\]

The optimality conditions for this problem is given as:

\[
-\sum_{w \in W} d_w \sum_{k \in K_w ; k \neq a} p_k^* + f_a(c_a^*) = 0 \ \forall a \in A,
\]

\[
\left( -c_w^* + \frac{1}{K_w} e^T c_w^* e \right) - \left( g(p_w^*) - \frac{1}{K_w} e^T g(p_w^*) e \right) = 0 \ \forall w \in W
\]

\[
p_w^* \in \Delta_w \ \forall w \in W,
\]

where:

\[
c_w = (c_k)_{k \in K_w} \text{ with } c_k = \sum_{a \in A ; k \neq a} c_a,
\]

and:

\[
g(p_w) = -\frac{1}{2} \left( \text{diag} \left( \Sigma_w^{1/2} \left( \Sigma_w^{1/2} S(p_w) \Sigma_w^{1/2} \right)^{1/2} \right)^T \Sigma_w^{1/2} \right) - 2 \Sigma_w^{1/2} \left( \left( \Sigma_w^{1/2} S(p_w) \Sigma_w^{1/2} \right)^{1/2} \right)^T \Sigma_w^{1/2} p_w.
\]

We now provide the formal definition of the CMM-SUE, where the equilibrium flows can be characterized as solutions that satisfy optimality conditions given in (2.16).

**Definition 2.2 (CMM-SUE)** In the CMM-SUE, the equilibrium arc flow is the solution to the fixed point equation:

\[
f_a = \sum_{w \in W} d_w \sum_{k \in K_w ; k \neq a} \mathbb{P}_{\theta_{cmm}} \left( k = \arg \max_{l \in K_w} (-c_l(f) + \bar{e}_l) \right) \ \forall a \in A,
\]

where the choice probability is computed for the distribution of \( \bar{e}_w = (\bar{e}_k)_{k \in K_w} \) with mean 0 and covariance matrix \( \Sigma_w > 0 \) that maximizes the expected user utility:

\[
\theta^*_{cmm} = \arg \max_{\bar{e}_w \sim \theta(0, \Sigma_w)} \mathbb{E}_{\theta} \left( \max_{k \in K_w} (-c_k(f) + \bar{e}_k) \right) \ \forall w \in W.
\]
Under the assumption that the matrix $\Sigma_w \succ 0$ for all origin-destination pairs $w \in W$ and the link travel costs are differentiable, positive and strictly increasing in their argument, the CMM-SUE equilibrium exists and is strictly unique. This follows from the observation that the objective function in (2.15) is strictly convex with respect to the $c$ variables and strictly concave with respect to the $p$ variables (see Proposition 1.1). In the next section, we develop an algorithm to compute CMM-SUE.

3 Solution Algorithm For CMM-SUE

One of the well-known algorithms to find the arc flows of the MNP-SUE formulation is the Method of Successive Averages (MSA) developed by Sheffi and Powell [43]. MSA is a gradient-based method where in each step the current iterate is updated by adding a multiple of a descent direction. Finding a descent direction is not straightforward because it involves calculating path probabilities. Instead of calculating exact descent directions in each iteration, Sheffi and Powell [43] propose to use an estimate direction obtained from a Monte-Carlo simulation. Even though every estimate may not be a descent direction, the average of the directions used is expected to be a descent direction. The algorithm converges to the optimal solution almost surely under mild conditions. Maher [30] developed link-based algorithms for the logit SUE problem, using Sheffi and Powell’s formulation. Maher proposed an algorithm that uses the same search direction as the MSA algorithm, but calculates an approximately optimal step size in this direction, thus improving overall convergence. Two adaptations of the Davidon-Fletcher-Powell (DFP) method were also considered, but were found inferior to the above method. Another example is the entropy-based algorithm developed by Dial, which is specific for the logit route choice model. This algorithm exploits the fact that for the logit function, it is possible to map path flows from link flows and vice-versa. Dial’s and Maher’s algorithms exploit mathematical properties of the logit function to develop efficient link-based algorithms. Recent research on path-based algorithms has demonstrated and established that it is a viable approach for deterministic traffic assignment problems with reasonably large network size (see, for example [18]). Most of the work has been on two algorithms: the Disaggregated Simplicial Decomposition (DSD) algorithm and the gradient projection (GP) algorithm. The adaptation of these algorithms to the logit SUE are discussed by Bekhor and Toledo [9, 23].

In this section, we propose a simple descent algorithm to compute the equilibrium arc flows for the CMM-SUE. The main difference between the MSA algorithm used for calculating the MNP-SUE and our algorithm is that we exploit optimization to find the choice probabilities rather than using simulation as is the case of MNP choice models. Therefore, we are able to find descent directions at all iterations of the algorithm. This guarantees that the algorithm converges to the equilibrium and the computational efficiency in each iteration is significantly improved.

As discussed in the last section, the CMM-SUE arc flows can be obtained from the optimization problem (2.15) which can be rewritten as:

$$\min_{c} \max_{p \in \Delta} \Phi(c, p) = \min_{f} \Phi_f(f),$$

in terms of the arc flow variables, where $\Phi_f(f) := \max_{p \in \Delta} \Phi(c(f), p)$. For fixed $f$, $\Phi_f(f)$ is separable in $|W|$ subproblems, each one of which is equivalent to finding the optimal choice probabilities in a CMM model, i.e., each subproblem corresponds to an origin-destination pair $w \in W$ and can be solved efficiently by applying Algorithm 1 with $\mu = -c_w$ and $\Sigma = \Sigma_w$.

Let $p^*(f) := \arg \max_{p \in \Delta} \Phi(c(f), p)$. The partial derivative of $\Phi_f(f)$ with respect to flow
variable $f_a$ is as follows:

$$\frac{\partial \Phi_f}{\partial f_a}(f) = -\frac{\partial c_a(f_a)}{\partial f_a} \left( \sum_{w \in \mathcal{W}} \sum_{k \in \mathcal{K}_w : k \ni a} d_w(p_w^*)_k - f_a \right).$$

Since $c_a$ is an increasing function of flow $f_a$, the vector

$$\left( \sum_{w \in \mathcal{W}} \sum_{k \in \mathcal{K}_w : k \ni a} d_w(p_w^*)_k - f_a \right)_{a \in A}$$

is a descent direction. This leads to a descent algorithm for solving $\min_f \Phi_f(f)$, which also calculates equilibrium flows for the CMM-SUE formulation.

**Input**: Parameters: Directed graph $\mathcal{G} = (\mathcal{N}, \mathcal{A})$, set of origin-destination pairs $\mathcal{W}$, demand $d_w$ and a set of paths $\mathcal{K}_w$, and covariance matrix $\Sigma_w \ \forall w \in \mathcal{W}$, cost function $c_a(f_a), \forall a \in \mathcal{A}$; Starting point: $f^0 = 0$; Tolerance: $\epsilon, \bar{\epsilon}$.

**Output**: Solution equilibrium arc flows $f^i$ for CMM-SUE.

**Algorithm 2**: CMM-SUE

In our numerical experiments, we use the following criterion to terminate the algorithm inspired from Sheffi [42]. Let $\bar{f}^i$ denote the flow average over the last $T$ iterations. Then we stop if

$$\frac{\|\bar{f}^i - f\|}{\|\bar{f}^i\|} \leq \bar{\epsilon}.$$

In our experiments, we set $T = 3$. Algorithm 2 is similar to the MSA algorithm with the main difference being in the calculation of the choice probabilities. Both algorithms show slow convergence properties due to their first-order nature and lack of an optimal step size. Nevertheless Algorithm 2 often outperforms MSA for MNP-SUE in terms of computational efficiency as we will demonstrate next.

4 Numerical Tests

4.1 A Simple Example with Two Links

When it comes to the computational efficiency, the advantage of using the CMM-SUE over the MNP-SUE is apparent even on the simplest graphs. We illustrate this fact on the two arc graph given in Figure 4.1. Assume that 1 unit of traffic flows from node $A$ to $B$. Travelers perceive the cost of the upper arc as $c_1(x) = x + \tilde{\epsilon}_1$ and the lower one as $c_2(x) = 1 + \tilde{\epsilon}_2$ as a function of flow variable $x$ on each arc. Further assume that $\text{Var}(\tilde{\epsilon}_i) = \sigma_i^2, i = 1, 2$, and $\text{Corr}(\tilde{\epsilon}_1, \tilde{\epsilon}_2) = \rho$. 

If we also assume that $[\tilde{\epsilon}_1, \tilde{\epsilon}_2]$ follows multivariate normal distribution, then the equilibrium arc flows for the MNP-SUE is implicitly given by:

$$x_{mnp}^* = \Phi \left( \frac{1 - x_{mnp}^*}{\sqrt{2\sigma^2(1 - \rho)}} \right), \quad (4.20)$$

where $x_{mnp}^*$ denotes the equilibrium arc flow on the upper arc, i.e., the choice probability of the upper arc with the MNP model, and $\Phi$ is the cumulative distribution function of the standard normal variable. While it is not possible to solve explicitly for $x_{mnp}^*$ from this equation, nevertheless, a close approximation can be found by using a simple bisection search method.

On the other hand, without the normality assumption, an argument similar to that in Mishra et al [32] provides that the arc flows in the CMM-SUE model is implicitly given as:

$$x_{cmm}^* = \frac{1}{2} \left( 1 + \frac{1 - x_{cmm}^*}{\sqrt{(1 - x_{cmm}^*)^2 + 2\sigma^2(1 - \rho)}} \right), \quad (4.21)$$

where $x_{cmm}^*$ denotes the equilibrium arc flow on the upper arc, i.e., the choice probability of the upper arc with the CMM model. We can obtain the value of $x_{cmm}^*$, given $\sigma$ and $\rho$, explicitly by solving a quartic equation. It is can be verified that as long as the two cost functions are linear functions of the flow variable, the equilibrium flow can be obtained from a quartic equation. There will be only one root that qualifies.

We provide the two equilibrium flows in Figure 4.2 for different values of $\sigma$ and $\rho \in [-1, 1]$. We observe that flows are quite close to each other in all cases, though the flow on the upper link in CMM-SUE tends to be higher than that in MNP-SUE. We also like to note that even with such a small example, calculating the MNP-SUE flows is five times slower than the CMM-SUE flows.

In the rest of this section, we compare the MNP-SUE and CMM-SUE for three transportation network examples discussed in the literature. The first example is a five-link road network taken from Daganzo [21], the second example is the classical Braess paradox network taken from Prashker and Bekhor [39], while the third example is the larger Sioux Falls network consisting of 24 nodes and 76 links. We compare the results of the MNP-SUE and CMM-SUE models in terms of the equilibrium flows and the computational times.

---

1See the Transportation Network Test Problems dataset at http://www.bgu.ac.il/~barga/tntp/
4.2 Small example from Daganzo [21]

The next example is a small network (Figure 4.3) taken from Daganzo [21]. This network has 5 links, 3 paths, and 100 units of flow between a single origin-destination pair. Therefore, it is simple enough to verify the MNP-SUE flows analytically as discussed by Daganzo [21]. We will find this fact useful in our discussion below. The deterministic costs associated with links are linear functions of the link flows as follows:

\[ c_1(f_1) = 7 + \frac{f_1}{22}, \quad c_2(f_2) = 5 + \frac{f_2}{78}, \quad c_3(f_3) = 5 + \frac{f_3}{78}, \quad c_4(f_4) = 7 + \frac{f_4}{22}, \quad c_5(f_5) = \frac{f_5}{56}. \]

The perceived cost for each link is the deterministic cost plus a random variable with zero mean and unit variance. It is assumed that these random variables are independent for each link.

We denote the set of paths as \( K = \{K_1, K_2, K_3\} \), where \( K_1 \) corresponds to the path following Link 1 and Link 2; \( K_2 \) to the path following Link 3 and Link 4; and path \( K_3 \) to the one following Link 3, Link 5, and Link 2. We also denote the choice probability of Path \( K_i \) as \( p_i \). Given a feasible flow \( f = [f_1, f_2, \ldots, f_5] \), the choice probabilities of the paths can be calculated from a choice model (either CMM or MNP) with mean \( \mathbf{c} = -[c_1(f_1) + c_2(f_2), c_3(f_3) + c_4(f_4), c_3(f_3) + c_5(f_5) + c_2(f_2)] \) and covariance matrix

\[
\Sigma = \begin{pmatrix}
2 & 0 & 1 \\
0 & 2 & 1 \\
1 & 1 & 3
\end{pmatrix}.
\]

As mentioned above briefly, we can verify that if the link flows are \( f_{\text{mnp}} = [22, 78, 78, 22, 56] \) and the link costs are \( c_{\text{mnp}} = [8, 6, 6, 8, 1] \), then the MNP-SUE conditions given in Equation (2.9) are satisfied. (See Daganzo [20] for the calculations of choice probabilities.)
Although we can easily verify that a given flow and a given cost vector together satisfy the MNP-SUE conditions, calculating the equilibrium flows from scratch is not straightforward. A table providing the flow values of the first 100 iterations of the MSA algorithm (which converges to MNP-SUE flows as discussed before) is provided in page 356 of [20]. It is observed that the flow values change rapidly in the beginning, but the convergence slows down after a while. After 100 iterations the flows are within 20 percent of the optimal values, but still far from the exact values. We use Algorithm 2 to calculate the CMM-SUE and present the results in Table 4.1 and 4.2. The first table provides the link flows, link costs, and the choice probabilities of paths after 40 iterations of the algorithm in the second column. We provide the exact MNP-SUE link flows, link costs, and path probabilities in the third column for comparison. We also provide the flow values over the 40 iterations of Algorithm 2 in more detail in the second table. The last row of this table also includes the flow values after 100 iterations of the MSA algorithm. The computational time needed for 40 iterations of Algorithm 2 is 0.46 seconds and for 100 iterations 0.66 seconds; on the other hand the MSA algorithm takes 6.3 seconds to perform 40 iterations and 13.5 seconds for 100 iterations. Interestingly, the CMM-SUE flows after a smaller number of iterations is closer to the exact MNP-SUE flows than those obtained from the MSA algorithm. Even though we start the two algorithms with the same initial flows, Algorithm 2 finds flows which are within 20 percent of the exact MNP-SUE flows in just few iterations, compared to the 100 iterations needed by the MSA algorithm. This is a result of the slow convergence of the MSA algorithm and the fact that CMM-SUE and MNP-SUE are relatively close to each other. We also observe that the CMM-SUE flows do not change much even if we let the algorithm run for a long time.

<table>
<thead>
<tr>
<th>Variables</th>
<th>CMM-SUE (approximate)</th>
<th>MNP-SUE (exact)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Link flow</td>
<td>(21.56, 78.44, 78.44, 21.56, 56.88)</td>
<td>(22, 78, 78, 22, 56)</td>
</tr>
<tr>
<td>Link cost</td>
<td>(7.980, 6.005, 6.005, 7.980, 1.015)</td>
<td>(8, 6, 6, 8, 1)</td>
</tr>
<tr>
<td>Path probability</td>
<td>(0.215, 0.215, 0.568)</td>
<td>(0.22, 0.22, 0.56)</td>
</tr>
<tr>
<td>Total travel cost</td>
<td>1344</td>
<td>1344</td>
</tr>
</tbody>
</table>

Table 4.1: Comparison of results between CMM and the MNP model

<table>
<thead>
<tr>
<th>Iteration</th>
<th>$f_1$</th>
<th>$f_2$</th>
<th>$f_3$</th>
<th>$f_4$</th>
<th>$f_5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>100</td>
<td>100</td>
<td>0</td>
</tr>
<tr>
<td>10</td>
<td>21.5277</td>
<td>78.2810</td>
<td>78.4723</td>
<td>21.7190</td>
<td>56.7532</td>
</tr>
<tr>
<td>20</td>
<td>21.5553</td>
<td>78.4089</td>
<td>78.4447</td>
<td>21.5911</td>
<td>56.8536</td>
</tr>
<tr>
<td>30</td>
<td>21.5596</td>
<td>78.4262</td>
<td>78.4404</td>
<td>21.5738</td>
<td>56.8666</td>
</tr>
<tr>
<td>35</td>
<td>21.5605</td>
<td>78.4295</td>
<td>78.4395</td>
<td>21.5705</td>
<td>56.8690</td>
</tr>
<tr>
<td>36</td>
<td>21.5606</td>
<td>78.4300</td>
<td>78.4394</td>
<td>21.5700</td>
<td>56.8693</td>
</tr>
<tr>
<td>37</td>
<td>21.5608</td>
<td>78.4304</td>
<td>78.4392</td>
<td>21.5696</td>
<td>56.8696</td>
</tr>
<tr>
<td>38</td>
<td>21.5609</td>
<td>78.4308</td>
<td>78.4391</td>
<td>21.5692</td>
<td>56.8699</td>
</tr>
<tr>
<td>39</td>
<td>21.5610</td>
<td>78.4312</td>
<td>78.4390</td>
<td>21.5688</td>
<td>56.8702</td>
</tr>
<tr>
<td>40</td>
<td>21.5611</td>
<td>78.4315</td>
<td>78.4389</td>
<td>21.5685</td>
<td>56.8704</td>
</tr>
<tr>
<td>Daganzo [21] (After 100 iterations)</td>
<td>21</td>
<td>79</td>
<td>79</td>
<td>21</td>
<td>58</td>
</tr>
<tr>
<td>Theoretical MNP</td>
<td>22</td>
<td>78</td>
<td>78</td>
<td>22</td>
<td>56</td>
</tr>
</tbody>
</table>

Table 4.2: Convergence comparison between CMM and the MNP model.

Given a feasible flow vector $f$, the total travel cost is calculated as

$$T(f) = \sum_a f_a c_a(f_a).$$
The system optimal (SO) solution is a flow vector, $f^*$, which minimizes the $T(f)$. The final row in Table 4.1 provides the total travel cost associated with the CMM-SUE and MNP-SUE flows. The travel costs are (almost for all practical purposes) equal for both equilibriums. In addition, the unique system optimal flow coincides with the exact MNP-SUE solution.

### 4.3 Braess Paradox Example

The Braess paradox occurs when adding extra capacity to a network, when the drivers choose their route selfishly, can in some cases increase overall travel time. This is because the Nash equilibrium of such a system is not necessarily optimal for the whole system. It is known that Braess paradox occurs with both linear and nonlinear cost functions for certain ranges of demand [see Prashker and Bekhor [8]].

In this section, we consider a simple network (given in Figure 4.4) with four nodes, five links, three paths, and a single origin-destination pair (nodes A and B). The link costs are nonlinear functions of the link flows (as in [8]) as follows:

$$
c_1 = 1 + 2(f_1/s_1)^2, \quad c_2 = 10 + (f_2/s_2)^2, \quad c_3 = 1 + 2(f_3/s_3)^2, \quad c_4 = 10 + (f_4/s_4)^2, \quad c_5 = 1 + 2(f_5/s_5)^2.
$$

The parameter $s_i$ captures the capacity of link $i$, which is set to $s_i = 3.2$, $\forall i$. 

![Figure 4.4: Braess Paradox Network.](image)

Our goal is to compare the total travel cost associated to various equilibrium flows with respect to the total travel cost of the system optimum flows. We consider the deterministic user equilibrium (UE), variants of the MNL-SUE with different parameters, the MNP-SUE, and the CMM-SUE. In addition, we also study the effect of removing Link 3 on the total travel cost under each equilibrium.

First consider the 4-link network, where Link 3 is removed. In this case, there are only two paths with symmetric travel costs. No matter how large the demand is, it would be distributed equally between the two paths in any (deterministic or stochastic) equilibrium, which is also the system optimal distribution. We capture the change in the total travel cost due to the existence of Link 3 under various equilibriums in Figure 4.5. The vertical axis provides the deviation of the total travel cost corresponding to different equilibrium flows from the optimal total travel cost on the network without Link 3. The total traffic flow in the network (total demand between the origin and the destination) is given on the horizontal axis. We observe that when the demand is low adding the extra link is beneficial. As the demand increases the Braess paradox is observed (for different ranges of demand) for all equilibrium solutions, i.e., the total travel cost with the extra link becomes higher than without it. The CMM-SUE and the MNP-SUE equilibriums behave quite close to each other, and they are more robust to the paradox than the UE equilibrium when the
demand is mild. When the network is very congested, i.e., the demand is high, the deterministic UE performs slightly better.

![Figure 4.5: Deviation from the 4-link solution.](image1)

We also compare the deterministic UE and SUE solutions with the SO solution on the original network. We measure the deviation of the total travel cost relative to the system optimal solution as:

\[
\frac{T(f) - T(f^{SO})}{T(f^{SO})},
\]

where \( f \) denotes the equilibrium flow we are referring to and \( f^{SO} \) denotes the system optimal flow. These results can be seen in Figure 4.6 again as a function of demand. The inflection point at 9.6 demand units suggests that the congestion effect is large enough to bring the UE and SO solutions closer to each other. For moderate congestion, the travel costs associated with all SUE flows are closer to the SO travel costs than the cost of the deterministic UE flows. The worst UE performance case is around 5 demand units with the highest deviation from optimal solution. The CMM-SUE flows are at par and at certain times outperform the ones calculated through either MNL or MNP models.

![Figure 4.6: Deviation from the optimal solution (5-link network).](image2)
4.4 Sioux Falls Network

In this section, we focus on a real transportation network, which is much larger than our previous examples. The aggregated network of the city of Sioux Falls, South Dakota, displayed in Figure 4.7, consists of 24 nodes, 76 links, and 552 origin-destination pairs. This network has been extensively used in the traffic equilibrium literature [see Morlok et al. [34], LeBlanc [26], LeBlanc et al. [28], Leblanc and Morlok [27] and Abdulaal and Leblanc [2]]. The link cost functions are in the following form:

\[ c_a = \bar{c}_a \left(1 + B \left(\frac{f_a}{\alpha_a}\right)^p\right), \]

where \( \bar{c}_a \) is the free flow travel cost. The related data such as link parameters of the cost functions and the demand between origin-destination pairs, collected by the Bureau of Public Research, can be found in detail in [34].

![Figure 4.7: Sioux Falls Network.](image)

On this large network we study three questions. The first test evaluates the effect of the step sizes (both in Algorithm 2 and the subroutine Algorithm 1) on the performance of the algorithm. The second test is conducted to understand the effect of the number of destination-origin pairs on the total time required to calculate the CMM-SUE and the MNP-SUE. In the last test, we measure how close the CMM-SUE and MNP-SUE equilibrium flows are to each other both in absolute values and also with respect to the associated total travel costs.

For the first test, we calculate the CMM-SUE equilibrium on this network for three origin-destination pairs using Algorithm 2. We set the stepsize used to three different levels, 0.02, 0.1, and 0.5. The algorithm calculates the choice probabilities by the subroutine given in Algorithm 1, we set the stepsize in this algorithm to the same three values as well. This gives us 9 different settings for the pair of stepsizes. We observed that there is no difference in the computational time required by the algorithm nor change in the equilibrium flows under different settings. This shows that the algorithm is robust against the choice of the stepsizes. Obviously, the performance of the algorithm in terms of the computational time could potentially be improved by calculating the optimal stepsize for each iteration, nevertheless, this is a cumbersome operation since this would require incorporating a line search in both algorithms.
We calculate the CMM-SUE and MNP-SUE flows on the network with seven different number of origin-destination pairs in the second test. When the number of OD pairs increases so does the complexity of the algorithm, due to the increase in the number of different paths. We report the computational time required by the corresponding algorithms in Table 4.3, where the number of OD pairs are given on the first column, the CPU times required by the two algorithms in the second and third column. We also report the difference times required by the two algorithms in the second and third column. We also report the difference between the total travel cost associated with the equilibrium flows (relative to the total cost of the MNP-SUE flow) on the forth column. The, so-called, relative difference is calculated as

\[ \frac{T(f^{CMM}) - T(f^{CMM})}{T(f^{MNP})}, \]

where \( f_{CMM} \) and \( f_{MNP} \) are the CMM-SUE and MNP-SUE flows, respectively.

<table>
<thead>
<tr>
<th>No. OD pairs</th>
<th>CMM-SUE</th>
<th>MNP-SUE</th>
<th>Relative difference</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>21.534</td>
<td>223.684</td>
<td>9.85E-02</td>
</tr>
<tr>
<td>20</td>
<td>32.237</td>
<td>306.636</td>
<td>6.45E-02</td>
</tr>
<tr>
<td>50</td>
<td>59.142</td>
<td>2419.863</td>
<td>7.60E-02</td>
</tr>
<tr>
<td>100</td>
<td>106.945</td>
<td>2804.022</td>
<td>1.03E-02</td>
</tr>
<tr>
<td>200</td>
<td>217.832</td>
<td>3332.762</td>
<td>1.18E-02</td>
</tr>
<tr>
<td>400</td>
<td>298.105</td>
<td>3436.769</td>
<td>1.50E-02</td>
</tr>
<tr>
<td>552</td>
<td>379.185</td>
<td>3516.276</td>
<td>1.90E-02</td>
</tr>
</tbody>
</table>

Table 4.3: Computation times (in seconds) required to calculate the CMM-SUE and the MNP-SUE flows.

We observe that the computational time required to calculate the CMM-SUE flows is an order of magnitude smaller than that of calculating the MNP-SUE flows. It seems that the increase in time with respect the number of origin-destination pairs is quite erratic for both methods, nevertheless, they seem to scale ‘somehow’ similarly as the ratio of the computational time required for the CMM-SUE flows to that of the MNP-SUE for 10, 20, or 552 OD pairs is around 1 to 10. This is obviously a very quick and rough conclusion, further tests are required for better understanding of the scalability of the two methods.

We also provide the CMM-SUE and MNP-SUE flows for 552 OD pairs using two figures. Figure 4.8 provides two graphs, corresponding to the two different equilibriums, where each link has width proportional to the flow on it. It can be seen that the link flow values of the two models are fairly similar. Actual values on all 76 links are provided in Figure 4.9 for a more refined comparison. This concludes our last test, as we have shown that both the total travel times and the equilibrium flows are fairly similar for both equilibrium models for this network. This may be a due to the fact that for 552 OD pairs, the network is fairly congested, and therefore both equilibrium flows are relatively close to the system optimal.

5 Conclusion

In this paper, we introduce a new stochastic user equilibrium that models the effect of congestion on the distribution of the traffic on the links of a network. Although it is similar as a concept to the existing stochastic user equilibrium models, the underlying route choice model is based on a relatively new distributionally robust discrete choice model that has not been considered in the literature before. The, so-called, cross moment model uses the mean and covariance information
Figure 4.8: The CMM-SUE and MNP-SUE flows.

Figure 4.9: CMM-SUE and MNP-SUE flows when there are 552 OD pairs.
related to the utilities of different alternatives without any further distributional assumptions. Therefore, it is richer than the logit based equilibrium models and more flexible than the probit based equilibrium model.

We show that the equilibrium flows are equal to the first order conditions of a minmax optimization problem. We propose a simple gradient descent algorithm to calculate the equilibrium flows and demonstrate that it is capable of attacking large networks efficiently. Using this method, we demonstrate on some small artificial networks as well as a larger network that the new equilibrium results in flows close to those from the probit based model and outperforms the deterministic user equilibrium when tested on a network with respect to the Braess paradox. In addition, the descent algorithm converges much faster than the similar methods for probit based models and robust to the choice of parameters of the algorithm such as the stepsizes.

We believe that the rich and robust nature of the model together with its computational tractability makes the proposed distributional robust equilibrium interesting for further study, especially on larger scale networks with added features such as multiple vehicle types and elastic demands.

References


