Probability Densities of Random Walks

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Outline

The random walk integrals

Definition

$$W_n(s) := \int_{[0,1]^n} \left| \sum_{k=1}^{n} e^{2\pi x_k i} \right|^s \mathrm{d}\boldsymbol{x}$$

for complex s. $W_n := W_n(1)$.

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- Work in progress. . .
- Makes heavy use of experimental mathematics.

"Had one not erred, one would have achieved less."

What we know

- $W_1(s) = 1$, $W_2(s) = \binom{s}{s/2}$. So $p_1(x) = \delta_1(x)$, $p_2(x) = \frac{2}{\pi\sqrt{4-x^2}}$.

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- p_n is unique as all moments are known and the interval of integration is finite.
- We shift focus from W_n to p_n, in particular p_3 and p_4.

Outline

1. Introduction
2. Expectations
   - Experimental maths 1
3. Densities
4. 3 and 4 steps
   - Experimental maths 2

Closed forms

Theorem (1), Borwein, Straub, W. (2009)

$$W_4(-1) = \frac{\pi}{4}\,{}_7F_6\left(\left.\begin{matrix} \frac{5}{4}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ \frac{1}{4}, 1, 1, 1, 1, 1 \end{matrix}\right| 1\right).$$

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Theorem (2), Borwein, Straub, W. (2010)

Both of the following are equal to $W_4(1)$*:*

$$\frac{3\pi}{4}{}_7F_6\left(\left.\begin{matrix}\frac{7}{4},\frac{3}{2},\frac{3}{2},\frac{3}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\\ \frac{3}{4},2,2,2,1,1\end{matrix}\right|1\right) - \frac{3\pi}{8}{}_7F_6\left(\left.\begin{matrix}\frac{7}{4},\frac{3}{2},\frac{3}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\\ \frac{3}{4},2,2,2,2,1\end{matrix}\right|1\right)$$

$$= \frac{9\pi}{4}{}_7F_6\left(\left.\begin{matrix}\frac{7}{4},\frac{3}{2},\frac{3}{2},\frac{3}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\\ \frac{3}{4},2,2,2,1,1\end{matrix}\right|1\right) - 2\pi\,{}_7F_6\left(\left.\begin{matrix}\frac{5}{4},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\\ \frac{1}{4},1,1,1,1,1\end{matrix}\right|1\right).$$

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Fear not! For we use the definition of Meijer G-functions to obtain the integrand for $W_4(-1)$:

$$\frac{\Gamma(\frac{1}{2}-t)^2\Gamma(t)^2}{\Gamma(\frac{1}{2}+t)^2\Gamma(1-t)^2}x^t = \frac{\Gamma(\frac{1}{2}-t)^2\Gamma(t)^4}{\Gamma(\frac{1}{2}+t)^2} \cdot \frac{\sin^2(\pi t)}{\pi^2}x^t,$$

using $\Gamma(t)\Gamma(1-t) = \pi/\sin(\pi t)$.

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We choose the contour to enclose the poles of $\Gamma(\frac{1}{2}-t)$. $\sin^2(\pi t)$ does not interfere with the residues, for it equals 1 at half integers, so it can be ignored. Then the right-hand side is the integrand of a $G^{2,4}_{4,4}$.

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However, $c := -G^{2,2}_{4,4}\left({0,1,1,1 \atop \frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2}}\middle|1\right)$ does. Experimentally we observed $a(1) = 4c$.

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We use these easy identities:

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(z^{-b_1}G^{2,2}_{4,4}\left(\begin{matrix}a_1,a_2,a_3,a_4\\ b_1,b_2,b_3,b_4\end{matrix}\middle|z\right)\right) = \frac{-1}{z^{1+b_1}}G^{2,2}_{4,4}\left(\begin{matrix}a_1,a_2,a_3,a_4\\ b_1+1,b_2,b_3,b_4\end{matrix}\middle|z\right)$$

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Using Nesterenko's theorem:

$$W_4(1) = \frac{4}{\pi^3} \int_0^1 \int_0^1 \int_0^1 \sqrt{\frac{x(1-y)(1-z)}{(1-x)yz(1-x(1-yz))}}\, \mathrm{d}x\mathrm{d}y\mathrm{d}z.$$

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Change of variable $z' = 1 - z$, then use $(z')^{\frac{1}{2}} = (z')^{-\frac{1}{2}}(1-(1-z')) = (z')^{-\frac{1}{2}} - (z')^{-\frac{1}{2}}(1-z')$ to split it into two integrals.

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Each integral satisfies *Zudilin's theorem*, which converts such integrals into ${}_7F_6$'s.

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When we pick the "right" integrals, the integrands (as functions of E and K) on both sides equal.

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- For $n = 2$ and 3 the probability is elementary.
- p_n is smooth for $n \geq 6$.

Lord Rayleigh

- Our definition of p_n takes advantage of radial symmetry. A true 2D probability density ψ_n requires

$$W_n(s) = \int_0^n \psi_n(x) x^s \; 2\pi x \mathrm{d}x.$$

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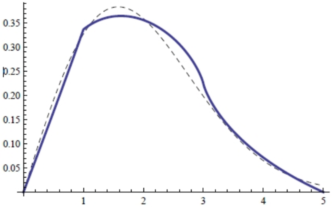
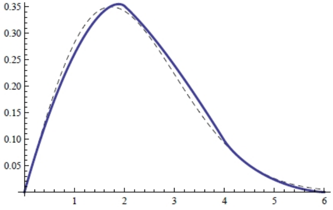
- $\psi_n(x) \approx \frac{1}{n\pi} e^{-x^2/n}$, like a 2D central limit theorem.
- This is very accurate even for moderate n.

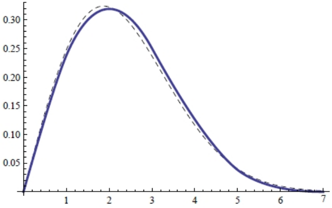
p_n with approximations superimposed.



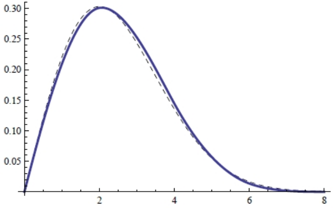
(a) p_5



(b) p_6



(c) p_7



(d) p_8

A better approximation is $xe^{-\frac{x^2}{n}}\left(\frac{4n^3-2n^2+4nx^2-x^4}{2n^4}\right)$.

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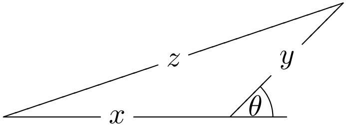

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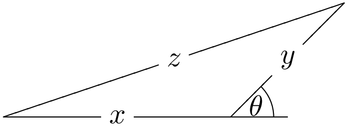

The moments are worked out by CAS:

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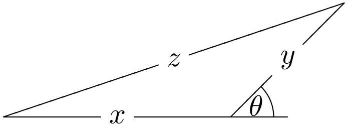

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Therefore $W_{n+m}(s) = \int_0^n \int_0^m g_s(x,y)\, p_n(x) p_m(y) \mathrm{d}x\mathrm{d}y.$ (1)

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$$p_3(x) = \frac{\sqrt{x}}{\pi^2}\ \mathrm{Re}\ K\left(\sqrt{\frac{(x+1)^3(3-x)}{16x}}\right).$$

Put $r=0$, we get $\psi_n(0) = \psi_{n-1}(1) = \frac{p_{n-1}(1)}{2\pi} = \frac{p_n'(0)}{2\pi} = \frac{\mathrm{Res}_{-2}W_n}{2\pi}$.

Alternative form for p_n

We now use the *sine rule* to make a change variable, so the last integral in (1) becomes $\mathrm{d}z$ instead of $\mathrm{d}y$:

$$W_{n+m}(s) = \int_0^{n+m} z^s \left\{ \int_0^n \int_0^\pi \frac{z}{\pi y} p_n(x) p_m(y) \mathrm{d}t \mathrm{d}x \right\} \mathrm{d}z,$$

where $y = \sqrt{x^2 + z^2 - 2xz\cos t}$.

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Combined with p_2, we have

$$p_4(t) = \frac{8t}{\pi^3} \int_0^2 \mathrm{Re}\left(\frac{K\left(\sqrt{\frac{16xt}{(x+t)^2(4-(x-t)^2)}}\right)}{\sqrt{(x+t)^2(4-(x-t)^2)}} \right) \frac{\mathrm{d}x}{\sqrt{4-x^2}},$$

which is better numerically than its Bessel counterpart.

Outline

Poles of W_3 via p_3

In p_3, we have $K\left(\sqrt{\frac{16x^3}{(3-x)^3(1+x)}}\right) = \frac{3-x}{3+3x}\,K\left(\sqrt{\frac{16x}{(3-x)(1+x)^3}}\right)$,
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So we can write p_3 cleanly in terms of the AGM, enabling us to use a result of Borwein et al. So on $[0,1)$

$$p_3(x) = \frac{2}{\sqrt{3}\pi} x \sum_{k=0}^{\infty} W_3(2k) \left(\frac{x}{3}\right)^{2k}.$$

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Using this series, we compute (with lots of care), for small $a > 0$,

$$\int_0^a p_3(x) x^s \mathrm{d}x = \frac{2a^{s+2}}{\sqrt{3}\pi(s+2)} + \frac{2a^{s+4}}{3\sqrt{3}\pi(s+4)} + \cdots$$

so the residues of W_3 can be read off, namely,

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But if p_4 admits a similar series, how can this reconcile with the double poles of W_4?

Functional equation for p_3

As $\operatorname{Re} K(x) = \frac{1}{x} K\left(\frac{1}{x}\right)$ for $x > 1$, we split p_3 over $[0,1]$ and $[1,3]$, obtaining $W_3(-1) = \int_0^3 \frac{p_3(x)}{x}\,\mathrm{d}x =$

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Also, $W_3(-1) = \frac{4}{\sqrt{3}\pi}\sum_{k=0}^{\infty} \frac{W_3(2k)}{9^k(2k+1)}$.

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Jon asked us to plot $p_4'(x)$ for small x. Armin correctly used the true formula,

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Unfazed by my failure to find a derivative from first principles, this means, very nearly, p_4 satisfies the differential equation

$$f'(x)+a=\frac{f(x)}{x},$$

which even I can solve: $f(x)=bx-ax\log x$, where $b \approx 0.33$ as $\int_0^1 f(x)\mathrm{d}x=\frac{1}{5}$.

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In fact, if the series were to be consistent with the residues and coefficients of the double pole, then we must have:

$$p_4(x) = \sum_{n=1}^{\infty} \big(a_4(n) - r_4(n) \log x\big) x^{2n-1},$$

where $a_4(n)$ are the residues at $-2n$ and $r_4(n)$ are the coefficients of the double pole at $-2n$.

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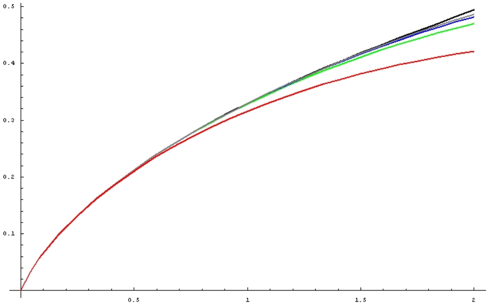
The first approximation is

$$\left(\frac{9 \log 2}{2\pi^2} - \frac{3}{2\pi^2} \log x\right) x.$$

$r_4(n)$ may be obtained in closed form by recursion.

p_4 versus conjectured expansion on $[0, 2]$.



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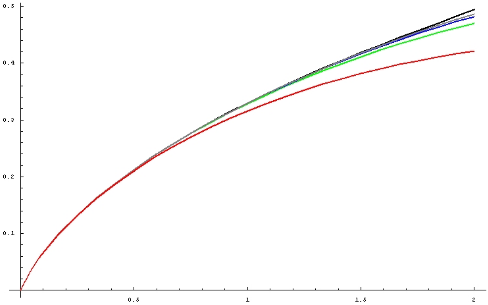

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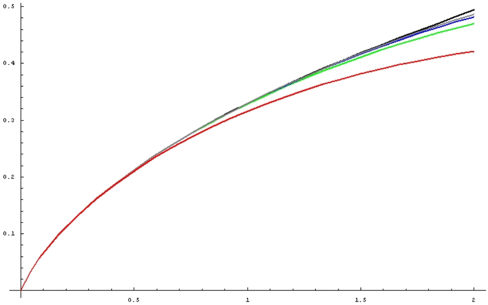

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p_4 can also be written in terms of the *Domb numbers*,

$$W_4(2n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k} \binom{2n-2k}{n-k}.$$

Closed forms

From our series for p_3, Zudilin (using modular tools) deduced the closed form

$$p_3(x) = \frac{2\sqrt{3}x}{\pi(3+x^2)}\ {}_2F_1\left(\left.{\frac{1}{3}, \frac{2}{3} \atop 1}\right| \frac{x^2(9-x^2)^2}{(3+x^2)^3}\right),$$

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as well as a closed formed for p_4 on $[2,4]$:

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Numerically, this works on $[0,4]$ by taking the real part.

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Numerically, this works on $[0,4]$ by taking the real part.

We get eerie connections with $W_3(s)$, for instance $p_4(2) = \frac{\sqrt{3}}{\pi}W_3(-1)$ and $p_3(\sqrt{3})^2 = 4p_3(2\sqrt{3}-3)^2 = \frac{3}{2\pi^2}W_3(-1)$.

Future work

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- Links to Calabi-Yau differential equations?
- More closed forms for derivatives and residues for W_3 and W_4.

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- Comments?
- Questions?
- Criticisms?