Densities of Uniform Random Walks on the Plane

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"Probabilists do it on random walks." – Anonymous

Co-authors: Jon Borwein, Dirk Nuyens, Armin Straub, and Wadim Zudilin

Outline

Random walk integrals

Definition: For complex s,

$$W_n(s) := \int_{[0,1]^n} \left| \sum_{k=1}^{n} e^{2\pi x_k i} \right|^s \mathrm{d}\boldsymbol{x}$$

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Makes heavy use of experimental mathematics (sophisticated playing).

"Had one not erred, one would have achieved less." – Euler

Computational challenge

- Dimension reduction:

$$W_n(s) = \int_{[0,1]^{n-1}} \left|1 + \sum_{k=1}^{n-1} e^{2\pi x_k i}\right|^s \mathrm{d}(x_1, \ldots, x_{n-1}).$$

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- $W_2(s) = \binom{s}{s/2}$, $W_2 = \frac{4}{\pi}$, $p_2(x) = \frac{2}{\pi\sqrt{4-x^2}}$.
- Tanh-sinh quadrature good for W_3, after splitting up region so 'bad' points lie on boundaries. Got 175 digits.

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- Binomial expansion:

$$W_n(s) = n^s \sum_{m \geqslant 0} \frac{(-1)^m}{n^{2m}} \binom{\frac{s}{2}}{m} \underbrace{\int_{[0,1]^n} \left(4 \sum_{i<j} \sin^2(\pi(x_j - x_i))\right)^m \mathrm{d}\boldsymbol{x}}_{=: I_{n,m}}$$

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- Recursion for $I_{n,m}$ gives accurate evaluations.

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- The recursion satisfies a convolution, which allows rapid computation of odd moments of n steps from $n-1$ steps (proven for many cases that concern us).
- We discover these relations because the *computer* allows us to see them.

Outline

Three steps

Theorem (1), Borwein, Nuyens, Straub, W. (2009)

$$W_3(1) = \frac{16\sqrt[3]{4}\pi^2}{\Gamma(\frac{1}{3})^6} + \frac{3\Gamma(\frac{1}{3})^6}{8\sqrt[3]{4}\pi^4},$$

$$W_3(-1) = \frac{3\sqrt[3]{2}}{16\pi^4}\Gamma^6\left(\frac{1}{3}\right).$$

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$$K(x) = \int_0^{\pi/2} \frac{\mathrm{d}x}{\sqrt{1 - x^2\sin^2(t)}}, \; E(x) = \int_0^{\pi/2} \sqrt{1 - x^2\sin^2(t)}\,\mathrm{d}x.$$

Four steps

Theorem (2), Borwein, Straub, W. (2009)

$$W_4(-1) = \frac{\pi}{4}\,{}_7F_6\left(\left.{\frac{5}{4}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \atop \frac{1}{4}, 1, 1, 1, 1, 1}\right| 1\right).$$

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Theorem (3), Borwein, Straub, W., Zudilin (2010)

$W_4(1) =$

$$\frac{3\pi}{4}\,{}_7F_6\left(\left.{\frac{7}{4}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \atop \frac{3}{4}, 2, 2, 2, 1, 1}\right| 1\right) - \frac{3\pi}{8}\,{}_7F_6\left(\left.{\frac{7}{4}, \frac{3}{2}, \frac{3}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \atop \frac{3}{4}, 2, 2, 2, 2, 1}\right| 1\right).$$

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Where the generalised hypergeometric series is

$${}_pF_q\left(\left.\begin{matrix}a_1, \ldots, a_p \\ b_1, \ldots, b_q\end{matrix}\right| z\right) = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{z^n}{n!}.$$

Theorem 3 first *identified* by *PSLQ*.

Cornelius Simon Meijer

Definition (Meijer G-functions)

$$G^{m,n}_{p,q}\left(\begin{matrix} a_1,\ldots,a_p \\ b_1,\ldots,b_q \end{matrix}\middle| x\right) := \frac{1}{2\pi i}\times$$

$$\int_L \frac{\prod_{j=1}^m \Gamma(b_j - s)\prod_{j=1}^n \Gamma(1-a_j+s)}{\prod_{j=n+1}^p \Gamma(a_j - s)\prod_{j=m+1}^q \Gamma(1-b_j+s)} x^s \mathrm{d}s.$$

The contour L lies between the poles of $\Gamma(1-a_i-s)$ and the poles of $\Gamma(b_i+s)$.

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Capture most special functions, very important in CAS, many definite integral evaluations are Meijer G transformations.

$$W_4(s) = \frac{2^s}{\pi}\frac{\Gamma(1+s/2)}{\Gamma(-s/2)} G^{2,2}_{4,4}\left(\begin{matrix}1,(1-s)/2,1,1\\ 1/2,-s/2,-s/2,-s/2\end{matrix}\middle| 1\right).$$

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- $a(z) := G^{2,2}_{4,4}\left({0,1,1,1 \atop -\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2}}\middle| z\right)$ not nice; $a(1) = -2\pi W_4(1)$.

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- Split the resulting triple integral into 2, each satisfies *Zudilin's theorem*: $\implies {}_7F_6$.

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- Theorem 2: transform to $G_{4,4}^{2,4}$, use *Bailey's identity* $\implies {}_7F_6$.
- *Nesterenko's theorem*: 'nice' $G_{4,4}^{2,4} \implies$ triple integral.
- $a(z) := G_{4,4}^{2,2}\left({0,1,1,1 \atop -\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2}}\middle|z\right)$ not nice; $a(1) = -2\pi W_4(1)$.
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Example of an easy Meijer G identity:

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(z^{-b_1}G_{4,4}^{2,2}\left(\begin{matrix}a_1,a_2,a_3,a_4\\b_1,b_2,b_3,b_4\end{matrix}\middle|z\right)\right) = \frac{-1}{z^{1+b_1}}G_{4,4}^{2,2}\left(\begin{matrix}a_1,a_2,a_3,a_4\\b_1+1,b_2,b_3,b_4\end{matrix}\middle|z\right).$$

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- So probability of returning to the unit disk is

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- p_n uses radial symmetry. A true 2D density ψ_n satisfies $p_n(x) = 2\pi x\psi_n(x)$.

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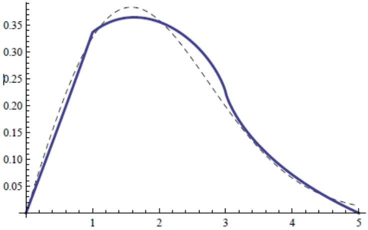
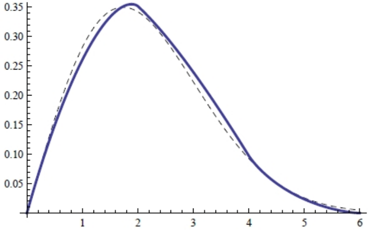
- $\psi_n(x) \approx \frac{1}{n\pi}e^{-x^2/n}$, like a 2D central limit theorem.

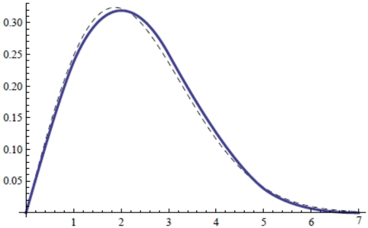
p_n with approximations superimposed.



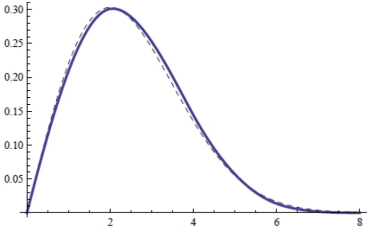
(a) p_5



(b) p_6



(c) p_7



(d) p_8

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- We were able to succeed by looking at moments and analytic structure.

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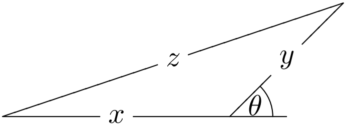
- By the *cosine rule*, $z^2 = x^2 + y^2 + 2xy\cos(\theta)$.



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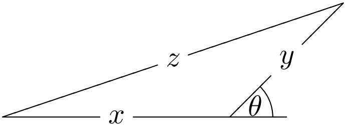

- Therefore
$$W_{n+m}(s) = \frac{1}{\pi}\int_0^n \int_0^m \left(\int_0^\pi z^s \mathrm{d}\theta\right) p_n(x)p_m(y)\mathrm{d}x\mathrm{d}y.$$

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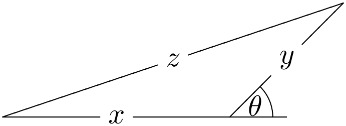

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- Change of variable: y as above,
$$W_{n+m}(s) = \int_0^{n+m} z^s \underbrace{\left\{\int_0^n \int_0^\pi \frac{z}{\pi y} p_n(x)p_m(y)\mathrm{d}t\mathrm{d}x\right\}}_{p_{n+m}} \mathrm{d}z.$$

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- $r = 0 : \psi_n(0) = \psi_{n-1}(1) = \frac{p_n'(0)}{2\pi} = \frac{\mathrm{Res}_{-2} W_n}{2\pi}.$

Outline

Closed form for p_3

- We experimentally found the *modular property*: with the involution $\sigma(x) = \frac{3-x}{1+x}$ (mapping $[0, 1)$ to $(1, 3]$),

$$p_3(x) = \frac{4x}{(3-x)(x+1)}\, p_3(\sigma(x)).$$

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- It follows that

$$p_3(x) = \frac{2x}{\sqrt{3}\pi} \sum_{k=0}^{\infty} W_3(2k) \left(\frac{x}{3}\right)^{2k}.$$

Poles of W_3 via p_3

- Using this series, we compute (taking care with convergence), for small $\alpha > 0$,

$$\int_0^\alpha p_3(x)x^s\mathrm{d}x = \frac{2\alpha^{s+2}}{\sqrt{3}\pi(s+2)} + \frac{2\alpha^{s+4}}{3\sqrt{3}\pi(s+4)} + \cdots$$

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- But $W_4(s)$ has double poles. It if p_4 admits a Taylor series around the origin, the above argument would produce simple poles. How to reconcile these two facts?

Series for p_4

- We wished to plot $p_4'(x)$ for small x. Best done from *first principles*. Instead of using the true formula,

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- which even I can solve: $f(x) = (a - r\log x)x$, where $a \approx 0.33$.

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- (This method generalises, and 'explains' the shape of p_5.)

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- DE rigorously produced by *Mellin transforms* of the recursion, *elliptic regularity theorem*, and a *Gosper*-like algorithm.
- More work on modular forms and hypergeometric series led to a closed formed for p_4 on $[0, 4]$:

Theorem (4) Borwein, Straub, W., Zudilin (2010)

$$p_4(x) = \frac{2\sqrt{16-x^2}}{\pi^2 x}\ \mathrm{Re}\ {}_3F_2\left(\begin{matrix}\frac{1}{2},\frac{1}{2},\frac{1}{2}\\ \frac{5}{6},\frac{7}{6}\end{matrix}\middle|\frac{(16-x^2)^3}{108x^4}\right).$$

Thank you!

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- Comments?
- Questions?
- Criticisms?