Legendre Polynomials and Ramanujan-type Series for $1/\pi$

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CARMA

Joint work with Heng Huat Chan and Wadim Zudilin

Outline

1 Introduction

2 Tools
- Ramanujan series
- Brafman's formula

3 Extensions
- Experimentation
- New series for $1/\pi$

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$$\sum_{n=0}^{\infty} \underbrace{\frac{(1/3)_n (2/3)_n}{n!^2}}_{binomial} (2+15n)\, P_n(3\sqrt{3}/5) \left(\frac{5}{6\sqrt{3}}\right)^n \stackrel{?}{=} \frac{45\sqrt{3}}{4\pi}.$$

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Hypergeometric series

$${}_pF_q\left(\begin{matrix} a_1,\ldots,a_p \\ b_1,\ldots,b_q \end{matrix}\middle| z\right) = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{z^n}{n!}.$$

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Simple ogf:

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Sun's series are of the form, for $s \in \{1/2, 1/3, 1/4, 1/6\}$,

$$\sum_{n=0}^{\infty} \frac{(s)_n (1-s)_n}{n!^2} (A + Bn) P_n(x_0) z_0^n = \frac{1}{\pi}.$$

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Notation

Introduce

$$F(t) = {}_2F_1\left(\begin{matrix} s, 1-s \\ 1 \end{matrix} \middle| t\right), \quad G(t) = t\,\frac{\mathrm{d}}{\mathrm{d}t} F(t).$$

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Clausen's formula (1828):

$${}_2F_1\left(\left.\begin{matrix} s,\ 1-s \\ 1 \end{matrix}\ \right|\ t\right)^2 = {}_3F_2\left(\left.\begin{matrix} \frac{1}{2},\ s,\ 1-s \\ 1,\ 1 \end{matrix}\ \right|\ 4t(1-t)\right).$$

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Then

$$F^2(t) = \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n (s)_n (1-s)_n}{n!^3} \big(4t(1-t)\big)^n,$$

$$F(t)G(t) = \frac{1-2t}{2(1-t)} \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n (s)_n (1-s)_n}{n!^3}\, n\big(4t(1-t)\big)^n.$$

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When τ is a quadratic irrational, $t(\tau)$ is *algebraic* and computable.

Series for $1/\pi$

Theorem (Ramanujan (1914), Borweins, Chudnovskys)

When α is t evaluated at a quadratic irrationality τ_0, there are computable algebraic constants a, b such that

$$\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n (s)_n (1-s)_n}{n!^3} (a+bn)\big(4\alpha(1-\alpha)\big)^n = \frac{1}{\pi}.$$

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Fred Brafman (1923 – 1959)

Brafman's formula (1951):

$$\sum_{n=0}^{\infty} \frac{(s)_n(1-s)_n}{n!^2} P_n(x) z^n = F\left(\frac{1-\rho-z}{2}\right) \cdot F\left(\frac{1-\rho+z}{2}\right),$$

where $\rho = (1 - 2xz + z^2)^{1/2}$.

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Proof via the elementary *Bailey's identity* (1933), which also gives Clausen's formula.

Let $\alpha = (1-\rho-z)/2, \beta = (1-\rho+z)/2$, then

$$\sum_{n=0}^{\infty} \frac{(s)_n(1-s)_n}{n!^2} P_n(x) z^n = F(\alpha)F(\beta),$$

$$\sum_{n=0}^{\infty} \frac{(s)_n(1-s)_n}{n!^2} n P_n(x) z^n = \frac{z(x-z-\rho)}{\rho(1-\rho-z)} G(\alpha)F(\beta) + \frac{z(x-z+\rho)}{\rho(1-\rho+z)} F(\alpha)G(\beta).$$

Computation of $F(\beta)$ and $G(\beta)$

Goal: reduce to expressions involving only $F(\alpha)$, $G(\alpha)$.

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$$\frac{1}{\alpha}\frac{\mathrm{d}\alpha}{\mathrm{d}\beta}\,G(\alpha) - \frac{1}{\beta}\,MG(\beta) = F(\beta)\frac{\mathrm{d}M}{\mathrm{d}\beta},$$

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Theorem (Chan, W., Zudilin (2011))

All of Sun's series involving P_n are true and we can produce arbitrarily many more.

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Theorem (Srivastava (1975), $N = 2$ case)

$$\frac{1}{\rho}\sum_{n=0}^{\infty}\frac{(1/2)_n^2}{n!^2}P_{2n}\left(\frac{x-z}{\rho}\right)\left(\frac{z}{\rho}\right)^{2n} = \sum_{n=0}^{\infty}u_n P_n(x)z^n,$$

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We need to generalise Brafman's formula, but first need a more powerful version of Bailey's identity.

Apéry-like sequences

Integer sequences (14 in total) that satisfy the recurrence

$$(n+1)^2 u_{n+1} = (an^2 + an + b)u_n - cn^2 u_{n-1}, \quad u_{-1} = 0,\ u_0 = 1.$$

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Inspired by the Clausen-type *ogf*^{2} of u_n, we *guessed*, with

$$g(X,Y) = \frac{X(1 - aY + cY^2)}{(1 - cXY)^2},$$

Experimental mathematics

Theorem (Generalised Bailey: W., Zudilin (2011))

$$\left\{\sum_{n=0}^{\infty} u_n X^n\right\}\left\{\sum_{n=0}^{\infty} u_n Y^n\right\} = \frac{1}{1-cXY}\sum_{n=0}^{\infty} u_n \sum_{m=0}^{n} \binom{n}{m}^2 g(X,Y)^m g(Y,X)^{n-m}.$$

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Generalized Brafman's formula

$$\sum_{n=0}^{\infty} u_n P_n\left(\frac{(X+Y)(1+cXY)-2aXY}{(Y-X)(1-cXY)}\right)\left(\frac{Y-X}{1-cXY}\right)^n$$
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With $s=\frac{1}{2}, K'(u)=\frac{\pi}{2}F(1-u^2)$

$$\sum_{n=0}^{\infty} \frac{(1/2)_n^2}{n!^2} P_{2n}\left(\frac{(u+v)(1-uv)}{(u-v)(1+uv)}\right)\left(\frac{u-v}{1+uv}\right)^{2n} = \frac{2(1+uv)}{\pi^2} K'(u)K'(v).$$

Proof by experimental mathematics

Let $H(x,y)$ be the double sum on the RHS and D_X be the annihilator of the ogf on the LHS.
We check that

$$(1-cXY)(D_X+D_Y)\big((H(x,y)/(1-cXY)\big) = \Delta_{x,y}H(x,y) = 0.$$

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Where

$$\Delta_{x,y} := (c(x^2+6xy+y^2)-a(x+y)+1)\left(x\,\frac{\partial^2}{\partial x^2}+y\,\frac{\partial^2}{\partial y^2}\right)$$
$$+4xy(2c(x+y)-a)\,\frac{\partial^2}{\partial x\,\partial y}+(c(5x^2+14xy+y^2)-a(3x+y)+1)\frac{\partial}{\partial x}$$
$$+(c(x^2+14xy+5y^2)-a(x+3y)+1)\frac{\partial}{\partial y}+2(c(x+y)-b).$$

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$$+ (c(x^2 + 14xy + 5y^2) - a(x + 3y) + 1) \frac{\partial}{\partial y} + 2(c(x + y) - b).$$

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$$+\,(c(x^2+14xy+5y^2) - a(x+3y)+1)\frac{\partial}{\partial y} + 2(c(x+y)-b).$$

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Equality follows from analyticity and form of D_X.

"Your brain is a liquid-cooled parallel supercomputer." – D. Chudnovsky

Exotic series

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$$\sum_{n=0}^{\infty}\left\{\sum_{k=0}^{n}\sum_{j=0}^{k}\binom{n}{k}\left(\frac{-1}{8}\right)^k\binom{k}{j}^3\right\} n\,P_n\left(\frac{5}{3\sqrt{3}}\right)\left(\frac{4}{3\sqrt{3}}\right)^n = \frac{9\sqrt{3}}{2\pi},$$

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Exotic series

Modularity is the key property for a $1/\pi$ series. We have proven new series involving binomial products times

$$P_n, P_{2n}, P_{3n}, \text{ and } u_n P_n.$$

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The first series, due to Ramanujan Himself, is equivalent to the second and the series on p3; all are *rational*.

Class number

$$\sum_{n=0}^{\infty} \frac{(1/4)_n(3/4)_n}{n!^2}(30282753 + 632736260n)P_n\left(\frac{561799}{73140\sqrt{59}}\right)\left(\frac{-6095\sqrt{59}}{821901696}\right)^n$$

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Thank you!

H. H. Chan, J. Wan and W. Zudilin
Legendre polynomials and Ramanujan-type series for $1/\pi$, *Israel Journal of Mathematics*, accepted.

Example of modular forms

Example: when $s = 1/2$, $F(t^2) = 2/\pi\, K(t) = \theta_3^2(q)$, where $t = \theta_2^2(q)/\theta_3^2(q)$, $q = e^{\pi i \tau}$.

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For $N = 2$, the quadratic transform $(1+t)K(t) = K(2\sqrt{t}/(1+t))$ corresponds to the *multiplier* $1/(1+t)$ and the degree 2 *modular equation* $\beta^2(\alpha+1)^2 - 4\alpha = 0$.

Example ($s = 1/3$)

$$\sum_{n=0}^{\infty} \frac{(\frac{1}{3})_n (\frac{2}{3})_n}{n!^2} (2 + 15n) P_n\left(\frac{3\sqrt{3}}{5}\right)\left(\frac{5}{6\sqrt{3}}\right)^n = \frac{45\sqrt{3}}{4\pi}.$$

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Here $\tau_0 = \sqrt{-1/3}$. $\alpha = 1/2 - 5/(6\sqrt{3}), \beta = 1/2$ satisfy the degree 2 modular equation

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$$M = \frac{F(\alpha)}{F(\beta)} = \frac{1}{2}\left(\frac{\beta^{2/3}}{\alpha^{1/3}} - \frac{(1-\beta)^{2/3}}{(1-\alpha)^{1/3}}\right),$$

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Taking derivative of the modular equation, $\mathrm{d}\alpha/\mathrm{d}\beta = 1/9$.

Substitute this into the derivative of the multiplier, we get

$$G(\beta) = \frac{2}{9}\,F(\alpha) + \frac{3\sqrt{3}+5}{3}\,G(\alpha).$$

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This is a multiple of

$$\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n(\frac{1}{3})_n(\frac{2}{3})_n}{n!^3}(2+15n)\left(\frac{2}{27}\right)^n = \frac{27}{4\pi},$$

due to Ramanujan.