Sums of Double Zeta Values

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Outline

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Split the second sum over the regions $m < n$, $m = n$, $m > n$, we get the key formula

$$\zeta(s)\zeta(t) = \zeta(s,t) + \zeta(t,s) + \zeta(s+t).$$

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$$(4)\ \sum_{j=1}^{s-1} (4^j + 4^{s-j})\zeta(2j, 2s-2j) = \left(s + \frac{4}{3} + \frac{4^s}{6}\right)\zeta(2s). \quad \text{(Nakamura 2009)}$$

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Formula (3) was originally proven using the closed form for $\zeta(n, 1)$ (from (1) and the key formula), together with induction on shuffle relations that arise from iterated integration of generalised polylogs.

Proof of (3)

Our aim is to find an easier proof. Write the LHS as

$$\sum_{j=2}^{s-1} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{2^j}{n^{s-j}(n+m)^j}.$$

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If $m=n$, we get $(s-2)\zeta(s)$. If $m\neq n$, do the geometric sum in j:

$$\sum_{m,n>0,\ m\neq n}\frac{2^s}{(n^2-m^2)(n+m)^{s-2}}-\frac{4}{(n^2-m^2)n^{s-2}}.$$

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$$\sum_{m>0,\ m\neq n}\frac{1}{m^2-n^2}=\frac{1}{2n}\sum_{m>0,\ m\neq n}\frac{1}{m-n}-\frac{1}{m+n}=\frac{3}{4n^2},$$

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Summing over n gives $3\zeta(s)$. We can all go home now.

Proof of (2)

Write out the sum in full and sum the geometric series first:

$$\sum_{m,n>0} \frac{1}{(m+n)(m+2n)n^{2s-2}} - \frac{1}{(m+2n)(m+n)^{2s-1}}.$$

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Let $k = m + n$:

$$\sum_{k>n>0} \frac{1}{k(k+n)n^{2s-2}} - \frac{1}{(k+n)k^{2s-1}}.$$

In the first term, use partial fractions and sum over k from $n+1$ to ∞; in the second term, sum over n from 1 to $k-1$. This gives:

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$$\left(\sum_{n>0} \frac{1}{n^{2s-1}} \sum_{k=n+1}^{2n} \frac{1}{k}\right) - \left(\sum_{k>0} \frac{1}{k^{2s-1}} \sum_{n=k+1}^{2k-1} \frac{1}{n}\right).$$

Now rename the variables in the second bracket, so the two sums telescope to $\sum_{n>0} 1/(2n^{2s}) = \zeta(2s)/2$.

Proof of (1)

Same deal: we again perform the geometric sum, then use partial fractions for the first equality, and telescoping for the second:

$$\begin{aligned}
&\sum_{m,n>0} \frac{1}{m(m+n)n^{s-2}} - \frac{1}{m(m+n)^{s-1}} \\
&= \sum_{n>0} \frac{1}{n^{s-1}} \sum_{m>0} \left(\frac{1}{m} - \frac{1}{m+n} \right) - \zeta(s-1,1) \\
&= \sum_{n=1}^{\infty} \frac{1}{n^{s-1}} \sum_{k=1}^{n} \frac{1}{k} - \zeta(s-1,1)
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&= \sum_{n=1}^{\infty} \frac{1}{n^{s-1}} \sum_{k=1}^{n} \frac{1}{k} - \zeta(s-1,1) \\
&= \sum_{n=1}^{\infty} \frac{1}{n^{s}} + \sum_{n=1}^{\infty} \frac{1}{n^{s-1}} \sum_{k=1}^{n-1} \frac{1}{k} - \zeta(s-1,1) \\
&= \zeta(s).
\end{aligned}$$

Comments

Take $s = 3$ in (1), we find $\zeta(2,1) = \zeta(3)$. Indeed, $\zeta(s,t)$ with $s + t \leq 6$ can be found this way. (Odd $s + t$ has closed form.)

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Adding/subtracting (1) and (3):

$$\sum_{j=1}^{s-1} \zeta(2j, 2s-2j) = \frac{3}{4}\zeta(2s), \quad \sum_{j=1}^{s-1} \zeta(2j+1, 2s-2j-1) = \frac{1}{4}\zeta(2s).$$

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The first equation and the key formula give

$$\sum_{j=1}^{s-1} \zeta(2j)\zeta(2s-2j) = \left(s + \frac{1}{2}\right)\zeta(2s).$$

Outline

Back to Riemann zeta and friends

$\zeta(2n)$ is the coefficient of x^{2n} in $f(x) = \frac{-\pi}{2} x \cot(\pi x)$ (contour integral of $f(x)x^{-2n-1}$). Writing the trig using exp, we have

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Similarly, tan, csc and sec give

$$\lambda(s) = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^s} = (1-2^{-s})\zeta(s),$$

$$\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} = (1-2^{1-s})\zeta(s),$$

$$\beta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)^s}.$$

Recursions of the zeta function

Via the key formula, any recursion of ζ or of B_n, of the form $\sum_j g(s,j)\zeta(2j)\zeta(2s-2j)$ gives a sum for the double zeta values.

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E.g. (4) comes from $\sum_{j=1}^{n-1} \binom{2n}{2j} 4^{-j} B_{2j} B_{2n-2j} = -\left(\frac{2n}{4^n} + 1\right) B_{2n}$.

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Using Ramanujan's formula

$$6\sum_{j=2}^{s-2}(2j-1)(2s-2j-1)\,\zeta(2j)\zeta(2s-2j) = (s-3)(4s^2-1)\zeta(2s),$$

we get another result due to Nakamura:

$$(5)\quad \sum_{j=2}^{s-2}(2j-1)(2s-2j-1)\,\zeta(2j,2s-2j) = \frac{3}{4}(s-3)\zeta(2s).$$

More recursions

With another of Ramanujan's results:

$$\sum_{j=1}^{n-1} j(2j+1)(n-j)(2n-2j+1)\,\zeta(2j+2)\zeta(2n-2j+2)$$

$$= \frac{1}{60}(n+1)(2n+3)(2n+5)(2n^2-5n+12)\zeta(2n+4) - \frac{\pi^4}{15}(2n-1)\zeta(2n),$$

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$$= \frac{3}{8}(n-1)(3n-2)\,\zeta(2n) - 3(2n-5)\,\zeta(4)\zeta(2n-4).$$

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We can also combine results of Miki and Matiyasevich:

$$\sum_{k=1}^{n-1}\left[1-\binom{2n}{2k}\right]\frac{B_{2k}B_{2n-2k}}{(2k)(2n-2k)} = \frac{H_{2n}}{n}B_{2n},$$

$$\sum_{k=1}^{n-2}\left[n-\binom{2n}{2k}\right]B_{2k}B_{2n-2k-2} = (n-1)(2n-1)B_{2n-2};$$

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Many recursions for B_n in the literature are reformulations of the zeta recursion by Euler and the first of Ramanujan's. There are few new ones (except those on the last slide, and some by Dilcher et al, Sun et al, etc). Also quite a few results have typos.

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Recall that $\beta(s)$ do not relate easily to $\zeta(s)$ and are expressible via the *Euler numbers* $\in \mathbb{Z}$, with egf $\frac{2e^t}{1+e^{2t}}$:

$$\beta(2n+1) = \frac{(-1)^n}{2(2n)!}\left(\frac{\pi}{2}\right)^{2n+1} E_{2n}.$$

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So we have recursions like

$$\sum_{k=1}^{n} \beta(2n+1-2k)\,\lambda(2k) = n\,\beta(2n+1).$$

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E.g. using ζ and η, we define

$$\zeta(s, \bar{t}) = \sum_{n=1}^{\infty} \sum_{m=1}^{n-1} \frac{1}{n^s} \frac{(-1)^{m-1}}{m^t},$$

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We get results for all character sums, e.g.

$$\begin{aligned} &2\sum_{j=1}^{s-1} \zeta(2j, \overline{2s-2j}) + \zeta(\overline{2j}, 2s-2j) \\ = &-4\sum_{j=1}^{s-1} \zeta(\overline{2j}, \overline{2s-2j}) = (1-4^{1-s})\zeta(2s). \end{aligned}$$

Outline

Why didn't we discover (3) earlier?

We know $\zeta(s,t)$ in closed form for $s+t$ odd or ≤ 7. We also assume that $\pi, \zeta(3), \zeta(5), \ldots$ are algebraically independent over $\mathbb{Q}$.

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So if we suspect a formula like $\sum_{j=2}^{s-1} a^j \zeta(j, s-j) = f(s)\zeta(s)$ exists, where f maps into $\mathbb{Q}$, then with $s=5$:

$$\frac{a^2}{6}\big(3(9-11a+4a^2)\zeta(5)-(a-1)(a-2)\pi^2\zeta(3)\big) = f(5)\zeta(5),$$

so $a=1$ or $a=2$.

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A similar experiment performed on $\sum_j (a \cdot b^j + c^s \cdot d^j)\zeta(j, s-j)$ $= f(s)\zeta(s)$, with rational a, b, c, d, summing over even/odd/all j, reveals that (1)–(4) are essentially the only possibilities.

Some experiments

We can also try to see if $\sum_j p(s,j)\zeta(j,s-j) = f(s)\zeta(s)$, where p is a 2-variable polynomial with rational coefficients. If deg $p \leq 2$, then (5) is the only possibility.

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A beautiful and deep generalisation of (1) is

$$\sum_{\sum_i a_i = s,\, a_i \ge 0} \zeta(a_1+2, a_2+1, \ldots, a_r+1) = \zeta(r+s+1).$$

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$W(r, s, 0) = \zeta(r)\zeta(s)$, $W(r, 0, t) = W(0, r, t) = \zeta(t, r)$. Easy to show $W(r, s, t) = W(r-1, s, t+1) + W(r, s-1, t+1)$, so W is expressible in terms of Riemann or double zeta values.

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Using our techniques, we have

$$(-1)^a \sum_{j=2}^{s-1} W(s-j,a,j) = \zeta(s+a) + \zeta(s+a-1,1) - (-1)^a \zeta(s-1,a+1)$$
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In particular, $\sum_{j=2}^{s}W(s-j,1,j)=\zeta(2,s-1)$, or equivalently

$$\sum_{j=2}^{s-1}j\,\zeta(j,s-j)=2\zeta(s)+\zeta(2,s-1)-(s-2)\zeta(s-1,1).$$

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The RHS has the difference of 2 character sums ($\zeta(\tilde{s},t),\ \zeta(\tilde{s},\tilde{t})$, where $\sim$ denotes the use of λ); evaluable for odd and for small s:

$$\sum_{j=2}^{s-1}2^{1-j}\zeta(j,s-j)=(2^{1-s}-1)\big(\zeta(s-1,1)-2\log(2)\zeta(s-1)\big)$$

$$+(2^{2-s}-1)\zeta(s)+\sum_{n=0}^{\infty}\frac{H_n}{(2n+1)^{s-1}}.$$

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By the way,

$$4\sum_{j=1}^{s-1}\zeta(\widetilde{2j},\widetilde{2s-2j}) = \sum_{j=1}^{s-1}\zeta(2j,\widetilde{2s-2j}) + \zeta(\widetilde{2j},2s-2j)$$

$$= (1-4^{-s})\zeta(2s).$$

Alternating double zeta values

The same elementary technique produces new relations such as

$$\sum_{j=1}^{s-1} \zeta(\bar{j}, s-j) = (2^{2-s}-1)\log(2)\zeta(s-1) - \zeta(\overline{s-1}, 1),$$

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Thank you!