Moments of Products of Elliptic Integrals

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Outline

Generalised hypergeometric functions

Recall that ${}_pF_q$ denotes the generalised hypergeometric series,

$$ {}_pF_q\left(\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix} \middle| z\right) = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{z^n}{n!}. $$

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They provide a framework for much of binomial sums, special functions, etc.

${}_2F_1$'s enjoy many transformations.

Complete elliptic integral K

Definition

The complete elliptic integral of the first kind is given by

$$\begin{aligned} K(x) &:= \int_0^{\pi/2} \frac{\mathrm{d}t}{\sqrt{1 - x^2 \sin^2 t}} \\ &= \int_0^1 \frac{\mathrm{d}t}{\sqrt{(1-t^2)(1-x^2t^2)}} \\ &= \frac{\pi}{2}\, {}_2F_1\left({\frac{1}{2}, \frac{1}{2} \atop 1} \middle| x^2 \right). \end{aligned}$$

So $K(1/\sqrt{2}) = \dfrac{\Gamma(1/4)^2}{4\sqrt{\pi}}$.

Complete elliptic integral E

Definition

The complete elliptic integral of the second kind is given by

$$\begin{aligned} E(x) &:= \int_0^{\pi/2} \sqrt{1 - x^2 \sin^2 t}\,\mathrm{d}t \\ &= \int_0^1 \frac{\sqrt{1-x^2t^2}}{\sqrt{1-t^2}}\,\mathrm{d}t \\ &= \frac{\pi}{2}\,{}_2F_1\left(\left.\begin{matrix} -\frac{1}{2}, \frac{1}{2} \\ 1 \end{matrix}\right| x^2\right). \end{aligned}$$

We let $x' := \sqrt{1-x^2}$, and $K'(x) := K(x')$, $E'(x) := E(x')$.

Basic properties

K and E are entangled by

$$\begin{aligned} \frac{\mathrm{d}E}{\mathrm{d}x} &= \frac{E-K}{x}, \\ \frac{\mathrm{d}K}{\mathrm{d}x} &= \frac{E-(1-x^2)K}{x(1-x^2)}, \\ \frac{\pi}{2} &= EK' + E'K - KK' \quad \text{(Legendre)}. \end{aligned}$$

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There is a third one, Π, but it is expressible as integrals of the first and second kinds.

Why do we care?

- Reason 1:

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- Perimeter of lemniscate ($r^2 = 2\cos 2t$) and ellipse.

They help us evaluate integrals

The area of an ellipse ($x^2 + y^2/b^2 = 1$) is very easy to find, but the perimeter is non-elementary.

$$P = \int_0^1 \sqrt{1 + (y')^2}\, \mathrm{d}x = \int_0^1 \sqrt{\frac{1 - (1 - b^2)x^2}{1 - x^2}}\, \mathrm{d}x = E'(b).$$

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The period of an ideal pendulum in a frictionless medium attached to a massless, inelastic string.

$$T'' + \frac{g}{L}\sin T = 0, \quad T \gg 0.$$

Period is $4\sqrt{\frac{L}{g}}K(\sin(a/2))$. When $a \to 0$, we recover $2\pi\sqrt{L/g}$. Note $g \approx \pi^2$, and also when $a = \pi$.

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Used to integrate $R(t, \sqrt{P(t)})$, where R is rational and P is cubic or quartic.

Digression on names

"The beginning of wisdom is to call things by their right names." $\overset{?}{-}$ *Confucius*

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But physicists also give a lot of "things" names. . .

> *"Many of us still feel more comfortable with a function if we have an explicit formula to look at. . . Some primitive people believe that if you know a man's name, then you have power over him. It is the same principle."* – *Underwood Dudley*

Nice connections

K and E have extremely important and nice properties of their own, and are very easy to compute.

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$$K(k) = \frac{\pi}{2}\theta_3^2(q), \quad \text{where } k = \frac{\theta_2(q)^2}{\theta_3(q)^2}.$$

$$\frac{K(k')}{K(k)} = -\frac{\log q}{\pi}.$$

Ties in with functional equations of θ; singular values, modularity and θ evaluations.

$$\theta_2(q) = \sum_n q^{(n+1/2)^2}, \ \theta_3(q) = \sum_n q^{n^2}.$$

Elliptic integrals are also inverses of elliptic functions.

Calculations

From (1),

$$\begin{aligned}\frac{\pi}{2}\frac{1}{K'(x)} &= M(1,x) = M\left(\frac{1+x}{2},\sqrt{x}\right) = \frac{1+x}{2}M\left(1,\frac{2\sqrt{x}}{1+x}\right)\\ &= \frac{\pi}{2}\frac{1+x}{2}\frac{1}{K((1-x)/(1+x))}.\end{aligned}$$

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$$K'(x) = \frac{2}{1+x}K\left(\frac{1-x}{1+x}\right), \quad (2)$$

$$K(x) = \frac{1}{1+x}K\left(\frac{2\sqrt{x}}{1+x}\right). \quad (3)$$

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(1) gives a fast way to calculate log. Thus, via Newton's method, all elementary functions can be computed efficiently via K.

Outline

Connections to random walks

Let $W_n(s)$ be the sth moment of the distance from the origin of an n step uniform random walk on the plane.

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Our research showed that

$$\begin{aligned} W_4(1) &= \frac{16}{\pi^3}\int_0^1 (1-3x^2)K'(x)^2\,\mathrm{d}x, \\ W_4(-1) &= \frac{4}{\pi^3}\int_0^1 K'(x)^2\,\mathrm{d}x. \end{aligned}$$

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$$2\int_0^1 K(x)^2\,\mathrm{d}x = \int_0^1 K'(x)^2\,\mathrm{d}x.$$

Now easily proven: set $x = (1-t)/(1+t)$ on the left, then use (2).

Organised search

David Bailey performed the following large scale search:

1. Given (d_1, d_2), compute integrals on $(0, 1)$ to 1500 digits and store in a list, using all integrands of the form

$$x^{i_0} K^{i_1} K'^{i_2} E^{i_3} E'^{i_4},$$

where $0 \leq i_0 \leq d_2$ and $i_1 + i_2 + i_3 + i_4 = d_1$.

2. Use integer relation program PSLQ on sets of $x = 40$ integrals, search for linear relations. When one is found, replace a linearly dependent integral by another in the list, repeat. Reduce x and keep finding relations. This produces a basis.
3. Write every integral in terms of the basis.

Example output

For $(d_1, d_2) = (2, 0)$:

K^2	KE	E^2	KK'	EK'	KE'		
1	0	0	0	0	0	-1	K^2
0	1	0	0	0	0	-1	KE
0	0	1	0	0	0	-1	E^2
0	0	0	1	0	0	-1	KK'
0	0	0	0	1	0	-1	EK'
-2	0	0	0	0	0	1	K'^2
0	0	0	0	0	-1	1	KE'
0	0	0	-1	2	1	-3	EE'
2	-4	3	0	0	0	-1	$K'E'$
6	-16	12	0	0	0	-3	E'^2

For $(d_1, d_2) = (2, 1)$:

K^2	xK^2	KE	xKE	KK'	xKK'	EK'	xEK'		
1	0	0	0	0	0	0	0	-1	K^2
0	1	0	0	0	0	0	0	-1	xK^2
0	0	1	0	0	0	0	0	-1	KE
0	0	0	1	0	0	0	0	-1	xKE
0	-2	2	4	0	0	0	0	-3	E^2
0	1	0	-3	0	0	0	0	2	xE^2
0	0	0	0	1	0	0	0	-1	KK'
0	0	0	0	0	1	0	0	-1	xKK'
0	0	0	0	0	0	1	0	-1	EK'
0	0	0	0	0	0	0	-1	1	xEK'
-2	0	0	0	0	0	0	0	1	K'^2
0	1	0	0	0	0	0	0	-1	xK'^2
0	0	0	0	-1	2	1	-4	1	KE'
0	0	0	0	0	0	0	-1	1	xKE'
0	0	0	0	0	-2	1	4	-3	EE'
0	0	0	0	0	1	0	-4	4	xEE'
-2	2	2	-4	0	0	0	0	1	$K'E'$
0	0	0	1	0	0	0	0	-1	$xK'E'$
-6	8	8	-16	0	0	0	0	3	E'^2
0	1	0	-3	0	0	0	0	2	xE'^2

For $(d_1, d_2) = (3, 0)$:

K^3	KE^2	E^3	KEK'	E^2K'	EK'^2		
1	0	0	0	0	0	-1	K^3
2	0	0	0	0	0	-3	K^2E
0	1	0	0	0	0	-1	KE^2
0	0	1	0	0	0	-1	E^3
10	0	0	0	0	0	-9	K^2K'
0	0	0	1	0	0	-1	KEK'
0	0	0	0	1	0	-1	E^2K'
-5	0	0	0	0	0	3	KK'^2
0	0	0	0	0	1	-1	EK'^2
-10	0	0	0	0	0	3	K'^3
10	0	0	-18	0	0	9	K^2E'
-115	192	-96	72	36	0	-72	KEE'
115	-192	96	0	-144	0	144	E^2E'
0	0	0	0	0	1	-2	$KK'E'$
-10	96	-48	0	0	-27	36	$EK'E'$
10	0	0	0	0	0	-9	K'^2E'
20	-48	24	0	0	-9	18	KE'^2
10	-60	30	0	0	9	-18	EE'^2
25	-64	32	0	0	0	-12	$K'E'^2$
-185	576	-288	0	0	0	48	E'^3

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Methodology

Elementary techniques:

1. Interchange order of summation and integration.
2. Change the variable x to x'.
3. Change the variable followed by quadratic transformations.
4. Use a Fourier series.
5. Apply Legendre's relation.
6. Differentiate then integrate by parts.

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With experimental tools:

PSLQ, ISC, OEIS, gfun, Gosper's algorithm, Sister Celine's method, Wolfram Functions. . .

Classical results

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$$\begin{aligned}
&\int_0^1 x^{u-1}(1-x)^{v-1}\,{}_2F_1\left(\begin{matrix}a, 1-a\\ b\end{matrix}\middle| x\right)\mathrm{d}x \\
= &\sum_{n=0}^{\infty}\frac{(a)_n(1-a)_n}{(b)_n\, n!}\int_0^1 x^{n+u-1}(1-x)^{v-1}\,\mathrm{d}x \\
= &\sum_{n=0}^{\infty}\frac{(a)_n(1-a)_n(u)_n}{(b)_n(u+v)_n\, n!}\frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)} \\
= &\frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)}\,{}_3F_2\left(\begin{matrix}a, 1-a, u\\ b, u+v\end{matrix}\middle| 1\right).
\end{aligned}$$

Closed forms

Hence,

$$\int_0^1 x'^n x^m K(x)\,\mathrm{d}x = \frac{\pi}{4}\frac{\Gamma(\frac{1}{2}(m+1))\Gamma(\frac{1}{2}(n+2))}{\Gamma(\frac{1}{2}(m+n+3))}\,{}_3F_2\left(\begin{matrix}\frac{1}{2}, \frac{1}{2}, \frac{m+1}{2} \\ 1, \frac{m+n+3}{2}\end{matrix}\middle|1\right),$$

$$\int_0^1 x'^n x^m E(x)\,\mathrm{d}x = \frac{\pi}{4}\frac{\Gamma(\frac{1}{2}(m+1))\Gamma(\frac{1}{2}(n+2))}{\Gamma(\frac{1}{2}(m+n+3))}\,{}_3F_2\left(\begin{matrix}-\frac{1}{2}, \frac{1}{2}, \frac{m+1}{2} \\ 1, \frac{m+n+3}{2}\end{matrix}\middle|1\right).$$

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By $x \mapsto x'$, we get moments of K' and E', which are in fact ${}_2F_1$'s and can be summed by Gauss' theorem.

Dixon's theorem applies to special ${}_3F_2$'s, e.g.

$$\int_0^1 x' E(x)\,\mathrm{d}x = \frac{1}{48\pi}\Gamma(1/4)^4.$$

Interchange order

To recap:

$$\int_0^1 x^n K'(x)\,\mathrm{d}x = \frac{\pi\,\Gamma(\frac{1}{2}(n+1))^2}{4\,\Gamma(\frac{1}{2}(n+2))^2},$$

$$\int_0^1 x^n E'(x)\,\mathrm{d}x = \frac{\pi\,\Gamma(\frac{1}{2}(n+3))^2}{2(n+1)\,\Gamma(\frac{1}{2}(n+2))\Gamma(\frac{1}{2}(n+4))}.$$

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$$K(x) = \sum_{k=0}^{\infty} \frac{\Gamma(k+1/2)^2}{\Gamma(k+1)^2}\frac{x^{2k}}{2},$$

$$E(x) = \sum_{k=0}^{\infty} -\frac{\Gamma(k-1/2)\Gamma(k+1/2)}{\Gamma(k+1)^2}\frac{x^{2k}}{4}.$$

So we can find $\int_0^1 x^n K(x)K'(x)\,\mathrm{d}x$ etc.

Closed forms

For instance,

$$\int_0^1 x^n K(x) K'(x)\,\mathrm{d}x = \frac{\pi^2}{8} \frac{\Gamma(\frac{1}{2}(n+1))^2}{\Gamma(\frac{1}{2}(n+2))^2}\, {}_4F_3\left({\frac{1}{2}, \frac{1}{2}, \frac{n+1}{2}, \frac{n+1}{2} \atop 1, \frac{n+2}{2}, \frac{n+2}{2}} \middle| 1 \right).$$

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For odd n, the nth moment of KK' is a rational multiple of π^3, and the nth moment of $K'E, KE', EE'$ is a rational multiple of $\pi^3 + \frac{\pi}{4(n+1)}$.

The second claim follows from the first and Legendre's relation.

For the first claim, let $g(n) := \int_0^1 x^{2n-1} K(x) K'(x)\,\mathrm{d}x$, then

$$2n^3 g(n+1) - (2n-1)(2n^2-2n+1) g(n) + 2(n-1)^3 g(n-1) = 0.$$

This contiguous relation can be proven using Gosper's algorithm.

Experiment

$h(n) := \pi^3 16^{n+1} g(n+1)$ matched entry A036917 of the OEIS (same recursion).
The OEIS tells us that

$$h(n) = \sum_{k=0}^{n} \binom{2n-2k}{n-k}^2 \binom{2k}{k}^2 = \frac{16^n \Gamma(n+1/2)^2}{\pi \Gamma(n+1)^2} \, {}_4F_3\left({{-n, -n, \frac{1}{2}, \frac{1}{2}} \atop {\frac{1}{2}-n, \frac{1}{2}-n, 1}} \middle| 1 \right).$$

The first equality is routine to check using Sister Celine's method.

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The ogf for $h(n)$ is simply

$$\sum_{n=0}^{\infty} h(n) t^n = \frac{4}{\pi^2} K(4\sqrt{t})^2,$$

again easy to prove using the series for K.

More products

Hence,

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It follows that

$$\begin{aligned} \int_0^1 \frac{2}{x} K(x)K'(x)(K(x)-E(x))\,\mathrm{d}x &= \int_0^1 K(x)^2 E'(x)\,\mathrm{d}x, \\ \int_0^1 \frac{-\log(1-x^2)}{x} K(x)K'(x)\,\mathrm{d}x &= \frac{7}{8}\pi\zeta(3). \end{aligned}$$

Tantalisingly close to the product of 3 integrals (missing a $\sqrt{}$).

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$$\begin{aligned}
\int_0^1 \frac{2}{x} K(x)K'(x)(K(x)-E(x))\,\mathrm{d}x &= \int_0^1 K(x)^2 E'(x)\,\mathrm{d}x, \\
\int_0^1 \frac{-\log(1-x^2)}{x} K(x)K'(x)\,\mathrm{d}x &= \frac{7}{8}\pi\zeta(3).
\end{aligned}$$

Tantalisingly close to the product of 3 integrals (missing a $\sqrt{}$).

There are 'meta'-reasons why the product of two elliptic integrals can be integrated (though hard), e.g. Mellin convolution, transform and Mellin-Barnes integral (Meijer G-function). It seems there is no equivalent for products of 3 functions (c.f. Bessel).

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$$\int_0^1\int_0^1\int_0^1 \frac{x^{h_2-1}y^{h_3-1}z^{h_4-1}(1-x)^{h_0-h_2-h_3}(1-y)^{h_0-h_3-h_4}(1-z)^{h_0-h_4-h_5}}{(1-x(1-y(1-z)))^{h_1}}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

$$= \frac{\Gamma(h_0+1)\prod_{j=2}^4\Gamma(h_j)\prod_{j=1}^4\Gamma(h_0+1-h_j-h_{j+1})}{\prod_{j=1}^5\Gamma(h_0+1-h_j)}\times$$

$${}_7F_6\left(\left.\begin{matrix} h_0, 1+h_0/2, h_1, h_2, h_3, h_4, h_5 \\ h_0/2, 1+h_0-h_1, 1+h_0-h_2, 1+h_0-h_3, 1+h_0-h4, h+h_0-h_5\end{matrix}\right|1\right).$$

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We use

$$\int_0^1 \frac{\mathrm{d}x}{\sqrt{x(1-x)(a-x)}} = \frac{2}{\sqrt{a}}K\left(\frac{1}{\sqrt{a}}\right), \quad \int_0^1\sqrt{\frac{a-x}{x(1-x)}}\mathrm{d}x = 2\sqrt{a}\,E\left(\frac{1}{\sqrt{a}}\right),$$

$$\int_a^1 \frac{\mathrm{d}y}{\sqrt{y(1-y)(y-a)}} = 2K'(\sqrt{a}), \quad \int_a^1\frac{\sqrt{y}}{\sqrt{(1-y)(y-a)}}\,\mathrm{d}y = 2E'(\sqrt{a}).$$

Manipulations

Using the above, we have, for instance,

$$\begin{aligned}
\int_0^1 E'(y)^2\,dy &= \frac{1}{2}\int_0^1\int_{a^2}^1 \sqrt{\frac{y}{(1-y)(y-a^2)}}E(\sqrt{1-a^2})\,\mathrm{d}a\mathrm{d}y \\
&= \frac{1}{4}\int_0^1\int_0^1 \sqrt{\frac{y}{(1-y)z(1-z)}}E(\sqrt{1-yz})\,\mathrm{d}y\mathrm{d}z \\
&= \frac{1}{8}\int_0^1\int_0^1\int_0^1 \sqrt{\frac{y(1-yz)}{(1-y)z(1-z)}}\sqrt{\frac{\frac{1}{1-yz}-x}{x(1-x)}}\,\mathrm{d}x\mathrm{d}y\mathrm{d}z \\
&= \frac{1}{8}\int_0^1\int_0^1\int_0^1 \sqrt{\frac{y(1-x(1-y(1-z)))}{x(1-x)(1-y)z(1-z)}}\,\mathrm{d}x\mathrm{d}y\mathrm{d}z.
\end{aligned}$$

The second equality comes from the change of variable $a^2 \mapsto yz$; the fourth from $z \mapsto 1-z$.

Closed forms

Now apply the theorem,

$$\int_0^1 x^n E'(x)^2 \, \mathrm{d}x = \frac{2^{4n}(n+1)^3(n+3)^2}{16(n+2)^3(n+4)} \frac{\Gamma\left(\frac{1}{2}(n+1)\right)^8}{\Gamma(n+1)^4}$$
$$\times \; {}_7F_6\left({-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{n+3}{2}, \frac{n+3}{2}, \frac{n+7}{4} \atop 1, \frac{n+3}{4}, \frac{n+2}{2}, \frac{n+4}{2}, \frac{n+4}{2}, \frac{n+6}{2}} \middle| 1 \right).$$

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This also gives all the odd moments of K^2, E^2, KE, by using

$$\int_0^1 x^{2n+1} K(x)^a E(x)^b K'(x)^c E'(x)^d \mathrm{d}x = \int_0^1 x(1-x^2)^n K'(x)^a E'(x)^b K(x)^c E(x)^d \mathrm{d}x.$$

E.g. $\int_0^1 xK'(x)^2 \mathrm{d}x = \frac{7}{4}\zeta(3)$, $\int_0^1 x^3 K(x)^2 \,\mathrm{d}x = \frac{1}{8}(2+7\zeta(3))$.

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When n is odd, the nth moment of the above functions is expressible as $a + b\zeta(3)$, $a, b \in \mathbb{Q}$ (via partial fractions).

Rant

If we write

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then $A_n = f(a), A_{n-1} = f'(a)/1!, \ldots, A_1 = f^{(n-1)}(a)/(n-1)!$.
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- Experimental maths.
- Turning up to seminars.

Beukers' integrals

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Our integrals also produce $\zeta(3)$, but the bound for the integrand is too poor.

Outline

Fourier series for K

$$K(\sin t) = \sum_{n=0}^{\infty} \frac{\Gamma(n+1/2)^2}{\Gamma(n+1)^2} \sin((4n+1)t).$$

Proof:

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- Transform the resulting ${}_3F_2$ and use Saalschütz's theorem.

Fourier series for E

Similarly (though not readily found in the literature),

$$E(\sin t) = \sum_{n=0}^{\infty} \frac{\Gamma(n+1/2)^2}{2\Gamma(n+1)^2} \sin((4n+1)t) + \sum_{n=0}^{\infty} \frac{(n+1/2)\Gamma(n+1/2)^2}{2(n+1)\Gamma(n+1)^2} \sin((4n+3)t).$$

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Using Parseval's theorem etc, integrals involving $K(\sin t)^2$ usually evaluated in terms of ${}_4F_3$'s.

However, using the odd moments,

$$\int_0^{\pi/2} K(\sin t)^2 \sin 4t \, \mathrm{d}t = -2.$$

Quadratic transform

Quadratic transformation (3) used on K'^n gives

$$\int_0^1 K'(x)^n \,\mathrm{d}x = 2 \int K(x)^n (1+x)^{n-2} \,\mathrm{d}x.$$

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Combining, we deduce

$$\int_0^1 K'(x)^3\mathrm{d}x = \frac{10}{3}\int_0^1 K(x)^3\mathrm{d}x = 5\int_0^1 xK(x)^3\mathrm{d}x = 5\int_0^1 xK'(x)^3\mathrm{d}x.$$

Legendre's relation

We can multiply Legendre's relation by a function and integrate:

$$\begin{aligned}
\int_0^1 3E'(x)K'(x)K(x) - K(x)K'(x)^2 \,\mathrm{d}x &= \frac{\pi^3}{8}, \\
\int_0^1 3E(x)K(x)K'(x) - 2K(x)^2K'(x) \,\mathrm{d}x &= \pi G, \\
\int_0^1 2E'(x)K(x)^2 - E(x)K(x)K'(x) \,\mathrm{d}x &= \pi G, \\
\int_0^1 2xE'(x)K(x)^2K'(x) - xK(x)^2K'(x)^2 \,\mathrm{d}x &= \frac{\pi^4}{32}.
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Unfortunately, we cannot uncouple these.

Integration by parts

$$\int_0^1 (1-x^2)^n \frac{\mathrm{d}}{\mathrm{d}x}\left(x^k K(x)^a E(x)^b K'(x)^c E'(x)^d\right) \mathrm{d}x$$
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E.g. using K'^2, we get $\int_0^1 2K'(x)E'(x)-(1-x^2)K'(x)^2\,\mathrm{d}x = 0$.

We obtain:

$$\begin{aligned}
\int_0^1 (n+2)x^{n-1}E(x)^2 - 2x^{n-1}E(x)K(x)\,\mathrm{d}x &= 1,\\
\int_0^1 \Big(nx^{n-1}E(x)K(x) - (n+2)x^{n+1}E(x)K(x) &+ \\
x^{n-1}E(x)^2 - x^{n-1}K(x)^2 + x^{n+1}K(x)^2\Big)\,\mathrm{d}x &= 0,\\
\int_0^1 2x^{n-1}E(x)K(x) + (n-2)x^{n-1}K(x)^2 - nx^{n+1}K(x)^2\,\mathrm{d}x &= 0.
\end{aligned}$$

Recursion

From these we obtain recursions: with $K_n := \int_0^1 x^n K(x)^2 \,\mathrm{d}x$,

$$(n+1)^3 K_{n+2} - 2n(n^2+1)K_n + (n-1)^3 K_{n-2} = 2.$$

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plus others.

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plus others.

Found by linear algebra or PSLQ. Give alternative proofs to $\pi^3, \zeta(3)$ results.

Towards the even moments

There are only five moments that we do not have closed forms of:

$$E(x)^2,\ x^2E(x)^2,\ E(x)K(x),\ x^2E(x)K(x),\ x^2K(x)^2,$$

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The last equation we need happens to be the only unproven entry from the $(2,0)$ and $(2,1)$ tables:

$$\int_0^1 2K(x)^2 - 4E(x)K(x) + 3E(x)^2 - K'(x)E'(x)\,\mathrm{d}x \stackrel{?}{=} 0.$$

Linearly related products

$$\begin{aligned}
\int_0^1 K(x)^2\,\mathrm{d}x &= \frac{1}{2}\int_0^1 K'(x)^2\,\mathrm{d}x \\
&= \int_0^1 K'(x)^2\,\frac{x}{x'}\,\mathrm{d}x \\
&= \int_0^1 K(x)K'(x)x'\,\mathrm{d}x \\
&= \int_0^1 \frac{2K(x)E(x)}{x+1}\,\mathrm{d}x \\
&= \frac{2}{\pi}\int_0^1 \frac{\arcsin x}{\sqrt{1-x^2}}K(x)K'(x)\,\mathrm{d}x \\
&= \frac{4}{\pi}\int_0^1 \operatorname{arctanh}(x)K(x)K'(x)\,\mathrm{d}x.
\end{aligned}$$

More related products

E.g. the derivative for $K(x)^3$ gives

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But we can't link the two groups.

Conjecture

Amazingly, the ISC gives

$$\int_0^1 K'(x)^3 \,\mathrm{d}x \overset{?}{=} 2K\left(\frac{1}{\sqrt{2}}\right)^4 = \frac{\Gamma(1/4)^8}{128\pi^2}.$$

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C.f. $\int_0^1 \frac{K'(x)}{\sqrt{1-t^2x^2}} \,\mathrm{d}x$, which at $t=1$ gives $K(1/\sqrt{2})^2$.

Equivalently,

$$\begin{aligned} &\sum_{n=0}^{\infty} \frac{8}{(2n+1)^2}\, {}_4F_3\left(\left.{\frac{1}{2}, \frac{1}{2}, n+1, n+1 \atop 1, n+\frac{3}{2}, n+\frac{3}{2}}\right| 1\right) \\ = \;& \sum_{n=0}^{\infty} \frac{\Gamma(n+1/2)^4}{\Gamma(n+1)^4}\, {}_4F_3\left(\left.{\frac{1}{2}, \frac{1}{2}, -n, -n \atop 1, \frac{1}{2}-n, \frac{1}{2}-n}\right| 1\right) \overset{?}{=} \frac{\Gamma(1/4)^8}{24\pi^4}. \end{aligned}$$

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Some sort of WZ method?

Conclusion

Many other relations proven using basic techniques.

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Thank you!