New Scaling of Child-Langmuir Law in the Quantum Regime

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This paper presents a consistent quantum mechanical model of Child-Langmuir (CL) law, including electron exchange-correlation interaction, electrode’s surface curvature, and finite emitter area. The classical value of the CL law is increased by a larger factor due to the electron tunneling through the space-charge potential, and the electron exchange-correlation interaction becomes important when the applied gap voltage $V_g$ and the gap spacing $D$ are, respectively, on the order of Hartree energy level, and nanometer scale. It is found that the classical scaling of $V_g^{3/2}$ and $D^{-2}$ is no longer valid in the quantum regime, and a new scaling of $V_g^{3/2} D^{-4}$ is established. The smooth transition from the classical regime to the quantum regime is also demonstrated.

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The classical Child-Langmuir (CL) law [1] gives the maximum current density allowed for steady-state electron beam transport across a gap of gap spacing $D$ and gap voltage $V_g$. In the one-dimensional (1D) planar, and non-relativistic model (with zero electron emission energy), the classical 1D CL law is

$$J_{CL} = \frac{4 \varepsilon_0}{9} \sqrt{\frac{2e V_g^{3/2}}{m D^2}}. \quad (1)$$

where $e$ and $m$ are the charge and mass of the electron, respectively, and $\varepsilon_0$ is the free space permittivity. From the equation, the classical scaling of the limiting current density $J_{CL}$ to the $3/2$’s power of gap voltage ($J_{CL} \propto V_g^{3/2}$), and to the inverse squared power of gap spacing ($J_{CL} \propto D^{-2}$), is widely used in the fields of high-current emission diodes, vacuum microelectronics, high power microwave sources, accelerator physics, and sheath physics. While Eq. (1) is easy to derive, it was only recently that the 1D classical CL law was extended to a two-dimensional (2D) model with simulation results [2], and with analytical solutions [3]. The effects of nonuniform current emission [4] from a finite emitter area and of the current’s short pulse [5] have also been studied. Recent developments of the multidimensional CL law in classical regime can be found in a review paper [6].

In the emerging fields of nanotechnology, nanostructures such as nanodiodes, nanotriodes, and nanogaps ranging from sub-10 nm to 100’s of nm are readily fabricated [7–9]. In such a nanometer scale, it is of interest to study the intense beam-gap interaction, such as the limiting current density (or CL law) for a nanosized gap, where quantum effects are important. However, compared to the classical models [1–6], there are relatively few investigations of CL law in the quantum regime [10,11]. In the 1D quantum models, mean field theory has been used to calculate the CL current density by assuming that the electron density in the gap is high enough to include only the electron space-charge field, but not for the electron quantum interaction (the exclusion principle is ignored) [10,11]. This assumption is clearly not valid when the electron’s energy scale is comparable to the Hartree energy, in which case the electron exchange-correlation interaction cannot be ignored.

Thus, in this context, several interesting questions arise: Is the electron exchange-correlation interaction important for a wide range of gap voltage and gap spacing? Are the classical scaling of $V_g^{3/2}$ and $D^{-2}$ valid in the quantum regime? What are the equivalent cylindrical and spherical CL laws in the quantum regime? How does one derive a simple 2D quantum CL law? This Letter addresses these questions.

Consider electrons with emission energy $E$ (with respect to Fermi energy, $E_F$) injected normally into a nanosize gap with a gap spacing $D$, where the emitter ($x = r_k$) is grounded, and the collector ($x = r_k + D$) is held at a dc voltage $V_g$. Using mean field theory, we solve the time-independent Schrödinger equation, the Poisson equation, and the charge conservation relation to obtain the mean electron density profile inside the gap for a given current density $J$. From the calculated density profile, we use the Kohn-Sham density functional theory [12] to calculate the electron exchange-correlation potential (in terms of the Hartree energy $E_H$):

$$V_{xc} = \varepsilon_{xc} = - \frac{\varepsilon_x}{3} \frac{d\varepsilon_{xc}}{dr_s}, \quad (2)$$

where $r_s$ is the local Seitz radius ($4\pi n r_s^3/3 = 1$), in terms of the Bohr radius $a_0$. Here, $\varepsilon_{xc} = \varepsilon_x + \varepsilon_c$ is the exchange-correlation energy per particle for a uniform electron gas of density $n$, under the Kohn-Sham local density approximation (LDA), where $\varepsilon_x$ is the exchange energy contribution [13] and $\varepsilon_c$ is the correlation energy
respectively, the normalized current density, normalized basic geometries: planar $(x = \frac{1}{D})$, $N = 0$, and $\phi = \phi(R+1) = 1$. The boundary conditions for Eqs. (4) and (5) are

$$\phi(R) = 0,$$  \hspace{1cm} (6a)

$$\phi(R+1) = 1,$$  \hspace{1cm} (6b)

$$q(R+1) = \frac{\sqrt{2\mu[N]R^N/3}}{(1 + \epsilon)^{1/4}(R+1)^{7/2}},$$  \hspace{1cm} (6c)

$$q'(R+1) = -(R+1)^{5/2}x[N]q(R+1),$$  \hspace{1cm} (6d)

where $\epsilon[N] = 0, \xi/\sqrt{R+1}$, and $(R+1)^2$ for $N = 0, 1$, and 2, respectively. Here, $\eta = (J_0 J_1 + Y_0 Y_1)/(J_0^2 + Y_0^2)$ and $J_0, J_1, Y_0, Y_1$ are the Bessel functions evaluated at the argument $\lambda(R+1)/\sqrt{1 + \epsilon}$. With the boundary conditions, we determine the quantum CL law, through the maximum value of $\mu[N]$, defined as $\phi_{\text{CL}}[N]$. For $\mu > \mu_{\text{CL}}$, solutions to Eqs. (4) and (5) no longer exist.

In the classical limit at $\lambda \gg 1$ and $\phi_{\text{g}} \gg 1$, Eqs. (4) and (5) become

$$\phi'' + \frac{N}{\lambda} \phi' = \frac{4}{9 \mu[N]} \frac{R^N}{\lambda^N \sqrt{x^N + \epsilon + \phi}},$$  \hspace{1cm} (7)

which is the governing equation (independent of $\lambda$ and $\phi_{\text{g}}$) to calculate the maximum value of $\mu_{\text{Q}}[N] = \mu_{\text{CL}}[N]$ in the classical regime. For a planar configuration ($N = 0$) at zero emission energy ($\epsilon = 0$), Eq. (7) simply gives $\mu_{\text{Q}} = 1$, which recovers the 1D classical CL law: $J_{\text{Q}} = J_{\text{CL}}[1]$.

Figure 1 shows the calculated $\mu_{\text{Q}}$ as a function of $\lambda$ for various values of $\phi_{\text{g}}$ at $\epsilon = 0$ and $N = 0$ (planar). For a given $\phi_{\text{g}}$, $\mu_{\text{Q}}$ increases with decreasing values of $\lambda$ (quantum regime) and exceeds the classically predicted values when $\lambda \approx 10$. The value of $\mu_{\text{Q}}$ is roughly proportional to $\lambda^{-2}$ at small $\lambda$, and this indicates that $J_{\text{Q}} \approx D^{-2}$ is the correct scaling in the quantum regime for small $D$, as compared to the classical scaling of $J_{\text{CL}} \propto D^{-2}$. From
the figure, we also see that the enhancement of $\mu_Q$ is more dominant at small $\phi_g$ due to electron exchange-correlation interaction, and previous results are recovered [10,11] at $\phi_g \gg 1$ (dashed line). This last statement implies that electron exchange-correlation interaction ($\phi_{xc}$) cannot be ignored at low gap voltage when it is on the order of Hartree energy level.

To investigate the scaling of limiting current to gap voltage in the quantum regime, the product of $\mu_Q$ and $V_g^{3/2}$ is plotted in Fig. 2 as a function of $V_g$ for various values of $D$ (solid lines). It is clear that the scaling of the limiting current density to the 3/2's power of gap voltage is no longer valid for small $V_g$ at a fixed $D$, and the new scaling is $I_Q \propto V_g^{1/2}$ in the quantum regime for small $V_g$.

The cases with no exchange-correlation interaction, $\phi_{xc} = 0$ (dashed lines), and the classical limit (short dashed line) are also plotted for comparison. The comparison proves the importance of the electron exchange-correlation interaction at small $V_g$ and $D$.

According to the classical CL law, the limiting current density ($\mu_Q \geq 1$) exists only for non-negative electron emission energy ($\epsilon \geq 0$). However, there is finite probability of electron tunneling in the quantum regime for negative electron emission energy ($\epsilon < 0$). In Fig. 3(a), we show $\mu_Q$ as a function of $D$ at $V_g = 1$ V for various values of $\epsilon = 0.5$ to $-0.5$. For $\epsilon < 0$, we see that $\mu_Q \rightarrow 0$ decreases sharply from the quantum regime to the classical regime with an increase of $D$. In Fig. 3(b), we show the boundary of the transition between the quantum and the classical regime at $E - E_F$ [eV] = $-0.2$, $-0.3$, and $-0.5$, where the boundary is defined as the calculated values of $D$ at $\mu_Q = 1$ for a given gap voltage $V_g$. For example, at $E - E_F = -0.2$ eV, the boundaries are $D = 63$ nm and $D = 240$ nm for $V_g = 1$ V and $10$ V, respectively.

In Fig. 4(a), we show the equivalent cylindrical ($N = 1$) and spherical ($N = 2$) models of the quantum CL law for $D = 10$ nm (solid lines) and $D = 100$ nm (dashed lines) at $\epsilon = 0$ and $V_g = 1$ V. For comparison, calculations at $D = 10$ nm and $V_g = 10$ V are also plotted, which show little differences between $V_g = 1$ V and $V_g = 10$ V. From the figure, we see that the limiting current $\mu [N = 1, 2]$ (with respect to $\mu [N = 0]$) increases with small values of $R$, and it recovers to the planar case at $R > 100$. Note that $\mu [N = 2]$ is larger than $\mu [N = 1]$ for a given $R$, $V_g$, and $D$, and that at a fixed value of $N$ the curves in Fig. 4(a) are insensitive to $D$ and $V_g$ over the range studied.

It has been shown recently that the 1D classical CL law can be extended analytically to a simple 2D model by using only the 1D electron density profile, and the analytical 2D classical CL law [3] agrees very well with simulation results [2]. Pretending that the same method may be applied to quantum corrections, we obtain the following expressions for $\mu_Q [2D]$:}

$$
\mu_Q [2D] = \frac{1 + \frac{4 D}{\pi W} \rho}{1 + \frac{D}{R} \rho}, \quad (8a)
$$

$$
\mu_Q [1D] = \frac{1 + \frac{D}{R} \rho}{1 + \frac{2}{\pi} \frac{D}{R} \rho}, \quad (8b)
$$

for an infinitely long emitting strip of width $W$, a circular emitting patch of radius $R$, and an emitting ellipse of semiaxes $R$ and $W/2 < R$, respectively. Here, the parameter $\rho$ measures the mean position of the electrons in the gap, and it is a function of $\lambda$ and $\phi_g$, defined as

$$
\rho(\lambda, \phi_g) = \int_0^1 \tilde{x} q^2(\tilde{x}) d\tilde{x} / \int_0^1 q^2(\tilde{x}) d\tilde{x},
$$

where $q^2(\tilde{x})$ is the

FIG. 2 (color online). The values of $\mu_Q V_g^{3/2}$ as a function of gap voltage $V_g$ (volt) for various gap spacing $D = 1$, 10, and 100 nm (solid lines: top to bottom). The dashed lines are without an exchange-correlation term ($\phi_{xc} = 0$) for $D = 10$ and 100 nm (top to bottom), and the short-dashed line is the classical limit.

FIG. 3 (color online). (a) The normalized quantum CL law $\mu_Q$ as a function of gap spacing $D$ (nm) for various values of $\epsilon = (E - E_F) / e V_g = 0.5$ to $-0.5$ at gap voltage $V_g = 1$ V. (b) The boundary of the transition from the quantum region (small $D$) to the classical region (large $D$) for $E - E_F$ [eV] = $-0.2$, $-0.3$, and $-0.5$. 208303-3
normalized 1D electron density profile calculated from solving Eqs. (4) and (5) at \( \mu = \mu_0(1D) \) at \( N = 0 \). Figure 4(b) gives the values of \( \rho \) as a function of gap voltage \( V_g \) for various values of gap spacing \( D = 10, 100, \) and \( 1000 \) nm. The classical limit is \( \rho = 0.25 \) (dashed line).

In conclusion, using mean field theory and LDA theory, a consistent and exact 1D quantum mechanical model of Child-Langmuir law, including the electron exclusion principle, is established for planar, cylindrical, and spherical geometries. For the first time, a new scaling of the Child-Langmuir law in the quantum regime, \( J_Q \propto V_g^{1/2} \) and \( J_Q \propto D^{-4} \), is established for small \( V_g \) and \( D \). The model provides an upper limit of electron current density for steady-state electron beam propagation in a nanosized gap, and it is independent of the nature of the electron emission process. A simple 2D quantum model is also developed based on the exact 1D model.

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[15] From Ref. [14], \( c = 0.75, \ A = 0.031091, \ a_1 = 0.082477, \ b_1 = 5.1486, \ b_2 = 1.6483, \ b_3 = 0.223647 \), and \( b_4 = 0.2064 \).