Grid Filter Modeling for Large-Eddy Simulation

Marc A. Habisreutinger, Roland Bouffanais, and Michel O. Deville

Abstract. An interpretation to the use of deconvolution models when used in implicitly filtered large-eddy simulations as a way to approximate the projective grid filter is given. Consequently, a new category of subgrid models, the grid filter models, is defined. This approach gives a theoretical justification to the use of deconvolution models without explicit filtering of the solution and explains how the use of such models can be effective in this context.

This viewpoint also allows to consider a new way of designing the convolution filter which has to approximate the grid filter and therefore a new way of improving such subgrid models. In this framework, a general technique for the approximation of the grid filter associated with any function-based numerical method is proposed. The resulting subgrid model is parameterless, only depends on the mesh used for the large-eddy simulation which is *a priori* known and vanishes locally if the flow is not turbulent, thereby ensuring the consistency of the model with the Navier–Stokes equations.

1 Introduction

In this work, the focus is put on large-eddy simulation (LES) based on deconvolution subgrid models which aim at a partial recovery of the full velocity field from

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its filtered counterpart by attempting to invert the filtering operator applied to the Navier–Stokes equations.

Deconvolution models are purely algorithmic as they only rely on the definition of the filter without the need to resort to any physical modeling. As noted by Domaradzki and Adams [1], "since the need for the physical models is removed this approach seems to be more promising than the classical models. However, this promise is not fulfilled if the effects of numerical discretization are not accounted for". Moreover, for a given filter, its inverse may be analytically determined which theoretically allows to express the full velocity field as a function of the filtered field. Therefore, solving the filtered Navier–Stokes equations should in principle deliver as much information as the solution of the standard non-filtered Navier–Stokes equations. This apparent paradox is resolved by acknowledging the unavoidable and irreversible effects of the implicit filter associated with the numerical discretization. This point has long been noted by LES practioners, *e.g.* by the earlier work of Zhou *et al.* [2], and more recently by Langford and Moser [3], Domaradzki and Loh [4], and Winckelmans *et al.* [5]. Nevertheless, the discretization effects are very often considered for the simple case of node-based methods but very few studies for function-based methods—*e.g.* spectral and finite/spectral element methods—are reported in the literature. Consequently, many of the arguments commonly expressed about grid filtering, are neither valid nor applicable when a function-based method is used as in the present study using the spectral element method (SEM).

When no explicit filter is applied but only the implicit grid filter is considered, deconvolution models can actively contribute to subgrid modeling but in a very different way than the one associated with LES relying on explicit filtering techniques. As a first step, an interpretation to the use of deconvolution models when used with implicit filtering as a way to approximate the projective grid filter is given. Consequently, a new category of subgrid models, the grid filter models (GFM) is defined. The GFM approach gives a theoretical justification to the use of deconvolution models without explicit filtering of the solution and explains how the use of such models can be effective in this context.

This viewpoint also allows to consider a new way of designing the convolution filter which has to approximate the grid filter and therefore a new way of improving such subgrid models. In this framework, a general technique for the approximation of the grid filter associated with any function-based numerical method is proposed, which addresses the previous note of Domaradzki and Adams.

2 Governing Equations

In the case of isothermal flows of Newtonian incompressible fluids, the LES governing equations for the filtered quantities denoted by an *overbar*, obtained by applying a convolution filter $\mathscr{G}\star$ to the Navier–Stokes equations, read

$$\frac{\partial \overline{u}_i}{\partial t} + \frac{\partial}{\partial x_j}(\overline{u}_i \overline{u}_j) = -\frac{\partial \overline{p}}{\partial x_i} + \nu \frac{\partial}{\partial x_j}\left(\frac{\partial \overline{u}_i}{\partial x_j} + \frac{\partial \overline{u}_j}{\partial x_i}\right) - \frac{\partial \tau_{ij}}{\partial x_j}, \tag{1}$$

$$\frac{\partial \overline{u}_j}{\partial x_j} = 0, \tag{2}$$

the filtered velocity field $\overline{\mathbf{u}} = \mathscr{G} \star \mathbf{u}$ satisfying the divergence-free condition (2) through the filtered reduced pressure field $\overline{p}$. The components of the subgrid tensor $\boldsymbol{\tau}$ are given by

$$\tau_{ij} = \overline{u_i u_j} - \overline{u}_i \overline{u}_j, \tag{3}$$

and ν is the kinematic viscosity. The closure of the filtered momentum equation (1) requires $\boldsymbol{\tau}$ to be expressed in terms of the filtered field which reflects the subgrid scales modeling and the interaction among all space scales of the solution.

The numerical method treats equations (1) and (2) within the weak Galerkin formulation framework. The spatial discretization relies on the SEM, see the monograph by Deville *et al.* [6] and Habisreutinger *et al.* [7] for full details.

3 Subgrid Modeling

The problem of subgrid modeling consists in taking into account the interaction between resolved and subgrid scales which is represented by the additional subgrid term in the filtered momentum equation (1) and thereby expressing the subgrid contribution as a function of the resolved quantities.

3.1 Deconvolution Models

The deconvolution approach aims at reconstructing the unfiltered fields from the filtered ones, the subgrid modes being not modeled but reconstructed using an *ad hoc* mathematical procedure. Borrowing the notation adopted in Habisreutinger *et al.* [7], the filtered Navier–Stokes momentum equation is written formally as

$$\frac{\partial \overline{\mathbf{u}}}{\partial t} + \mathbf{f}(\overline{\mathbf{u}}) = [\mathbf{f}, \mathscr{G}\star](\mathbf{u}), \tag{4}$$

where $[\mathbf{f}, \mathscr{G}\star](\mathbf{u})$ is the subgrid commutator, which only retains the nonlinear and non-commutating terms. This equation is strictly equivalent to (1) with

$$[\mathbf{f}, \mathscr{G}\star](\mathbf{u}) = -\nabla \cdot \boldsymbol{\tau}. \tag{5}$$

The exact subgrid contribution appears as a function of the non-filtered field, which is not computed when performing a LES. This field being unknown, the idea is to approximate it using a deconvolution procedure. For instance, Stolz & Adams [8] proposed the following procedure

$$\mathbf{u} \simeq \mathbf{u}^* = \mathscr{Q}_N \star \overline{\mathbf{u}} = (\mathscr{Q}_N \circ \mathscr{G}) \star \mathbf{u}, \tag{6}$$

where $\mathscr{Q}_N$ is an Nth-order approximation of the inverse of the filter $\mathscr{G}\star$ such as $\mathscr{Q}_N \circ \mathscr{G} = \mathscr{I} + O(\overline{\Delta}^N)$, $\overline{\Delta}$ being the filter cutoff length and $\mathscr{I}\star$ the identity filtering operator. The subgrid term is then approximated as

$$[\mathbf{f},\mathscr{G}\star](\mathbf{u}) \simeq [\mathbf{f},\mathscr{G}\star](\mathscr{Q}_N \star \overline{\mathbf{u}}). \tag{7}$$

3.2 Grid Filter Models

The previous developments do not take explicitly into account the major role played by the grid filter in LES. Indeed, the filter $\mathscr{G}\star$ to consider in practical simulations is a composition of the convolution filter $\mathscr{L}\star$ and the projective grid filter, referred to as $\mathscr{P}\star$ [9, 10] and represented by a *hat* in the sequel, such that

$$\mathscr{G}\star = (\mathscr{L} \circ \mathscr{P})\star. \tag{8}$$

Considering the grid filter in the filtered Navier–Stokes equations leads to

$$\frac{\partial \overline{\boldsymbol{u}}}{\partial t} + \mathbf{f}(\overline{\mathbf{u}}) = [\mathbf{f},(\mathscr{L} \circ \mathscr{P})\star](\mathbf{u}). \tag{9}$$

In the sequel, the focus is put on the particular case where implicit grid filtering is the only effective filter. The objective is to give a theoretical interpretation allowing to use a deconvolution model in this framework. When no LES filter $\mathscr{L}\star$ is explicitly applied, the filtered Navier–Stokes equations reduce to

$$\frac{\partial \hat{\mathbf{u}}}{\partial t} + \mathbf{f}(\hat{\mathbf{u}}) = [\mathbf{f},\mathscr{P}\star](\mathbf{u}), \tag{10}$$

where $\hat{\mathbf{u}}$ is the grid filtered velocity, or in other words the part of the velocity field that can be resolved by the grid used to perform the LES. In this framework, subgrid modeling based on deconvolution models requires the subgrid commutator of equation (10) to be expressed solely in terms of the known projected velocity $\hat{\mathbf{u}}$, which formally reads

$$[\mathbf{f},\mathscr{P}\star](\mathbf{u}) = [\mathbf{f},\mathscr{P}\star](\mathscr{P}^{-1}\star\hat{\mathbf{u}}), \tag{11}$$

where $\mathbf{u} = \mathscr{P}^{-1}\star\hat{\mathbf{u}}$ is the formal inverse grid filtering operation to be devised. It is worth noticing that due to its implicit nature, the grid filter $\mathscr{P}\star$ entirely depends on the numerical method used to discretize in space the Navier–Stokes equations.

The problem with equation(11) is that $\mathscr{P}^{-1}\star$ does not exist, this filter being a projector. The central idea introduced in Habisreutinger [11] and subsequently in Bouffanais [12] is then to approximate $\mathscr{P}\star$ with an invertible filter reproducing as closely as possible its effect

$$\mathscr{P}\star \simeq \mathscr{M}\star, \tag{12}$$

in order to have

$$[\mathbf{f},\mathscr{P}\star](\mathbf{u}) \simeq [\mathbf{f},\mathscr{M}\star](\mathscr{M}^{-1}\star\hat{\mathbf{u}}) = [\mathbf{f},\mathscr{M}\star](\mathbf{u}^\bullet), \tag{13}$$

with $\mathbf{u}^\bullet = \mathscr{M}^{-1} \star \hat{\mathbf{u}}$. For such subgrid models, modeling only requires to build $\mathscr{M}\star$ in order to achieve the best approximation of $\mathscr{P}\star$, as stated in equation (12). The problem is exactly solved if $\mathscr{P}\star = \mathscr{M}\star$, which is not achievable since $\mathscr{P}\star$ applies on an infinite spectrum while $\mathscr{M}\star$ only applies to computable wave numbers [1]. Moreover, $\mathscr{M}\star$ must be invertible while $\mathscr{P}\star$ is projective. The approximation (12) being the only required modeling, the subgrid models arising from this new approach are referred to as grid filter models. The governing equations to be solved in the GFM framework are obtained by combining equation (10) with approximation (13), and read

$$\frac{\partial \hat{\mathbf{u}}}{\partial t} + \mathbf{f}(\hat{\mathbf{u}}) = [\mathbf{f}, \mathscr{M}\star](\mathbf{u}^\bullet). \tag{14}$$

3.3 Grid Filter Modeling

As described in section 3.2, the GFM approach only needs to build an approximation of the grid filter in order to model the effect of the subgrid scales. This approximation can be performed in many different ways and from many different viewpoints which opens a full field of research.

In this work, the grid filter is approximated through a transfer matrix in a modal basis which was proposed in the p-version of finite elements and first used by Boyd [13] as a filtering technique. The transfer matrix is build in oder to preserve C^0-continuity across the elements, to be invertible and to approximate the grid filter. Its construction relies on an exact determination of the projection operation from the continuous space onto the LES mesh and on a statistical approximation of the unknown subgrid field which is based on a reciprocal probabilistic existence between every pair of modes of the spectrum. Consequently, the resulting subgrid model is parameterless, only depends on the LES mesh which is *a priori* known and vanishes locally if the flow is not turbulent, thereby ensuring the consistency of the model with the Navier–Stokes equations.

4 Lid-Driven Cavity Flow Simulation

A DNS of the flow in a lid-driven cubical cavity performed at Reynolds number of 12’000 with a Chebyshev collocation method due to Leriche and Gavrilakis [14] is taken as the reference solution to validate this new modeling approach. Subgrid modeling in the case of a flow with coexisting laminar, transitional and turbulent zones such as the lid-driven cubical cavity flow represents a challenging problem. As the flow is confined and recirculating, any under- or over-dissipative character of the subgrid model can be clearly identified. Moreover, the very low dissipation and dispersion induced by SEM allow a pertinent analysis of the energetic action induced by any subgrid model, which is not feasible in the framework of low-order numerical methods. The coupling of the lid-driven cubical cavity flow problem with the SEM builds therefore a well suited framework to analyze the accuracy of the newly defined subgrid model.

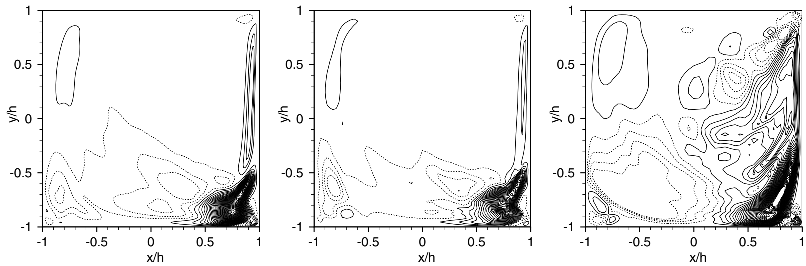

Fig. 1 Contours of $\langle \hat{u}_x^\circ \hat{u}_y^\circ \rangle$ from -0.0007 to 0.0065 in U_0^2 units, by increments of 0.0002 in the midplane $z/h = 0$. DNS (left), GFM (center), ILES (right). Dashed contours correspond to negative levels. The cavity lid $y/h = 1$, moves at U_0 in the x-direction

In an attempt to provide a comprehensive assessment of the performances of the GFM approach, the determination of the Reynolds stress tensor components has been envisaged as a challenge in the framework of the lid-driven cubical cavity flow. The Reynolds statistical decomposition $\mathbf{u} = \langle \mathbf{u} \rangle + \mathbf{u}^\circ$, introduces the average value denoted into brackets and its fluctuating part $\mathbf{u}^\circ$. The statistics for all LES are based on a sampling approximately 10 times smaller than the one of the DNS, more precisely 400 samples are collected over 80 time units.

Figure 1 displays the low-amplitude cross term $\langle \hat{u}_x^\circ \hat{u}_y^\circ \rangle$ in the symmetry plane of the cavity. The contours for Implicit LES (ILES) show that implicit modeling is totally inoperative in the SEM framework and highlight the need for explicit subgrid modeling as well as the under-resolution of the flow. The contours for GFM are relatively close to the DNS ones taking into account the reduced sampling and the low amplitude of the field as compared to the lid velocity $U_0 = 1$. This allows to asses the validity of this new modeling approach. However, further investigations and researches are required to improve or modify the grid filter modeling strategy briefly described in section 3.3, and to evaluate the relative performance of this models as compared to existing ones.

5 Conclusion and Perspectives

In conclusion, the GFM approach gives a theoretical justification to the use of deconvolution models without explicit filtering of the solution and explains how the use of such models works in this context. This viewpoint allows to consider a new way of designing the convolution filter which has to approximate the grid filter and therefore a new way of improving such subgrid models.

The results obtained so far demonstrate the validity of this modeling approach but further investigations are required to improve the grid filter modeling and to compare this models with existing ones.

Since the need for the physical models is removed, the validity of the GFM approach extends beyond the limited scope of incompressible Newtonian fluid flows considered in this article. For example, LES of compressible and viscoelastic fluid flows can also be envisaged.

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